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THE EQUATIONS OF MOTION OF AN AIRCRAFT EMBRACING ITS WHOLE-BODY--ETC(U)
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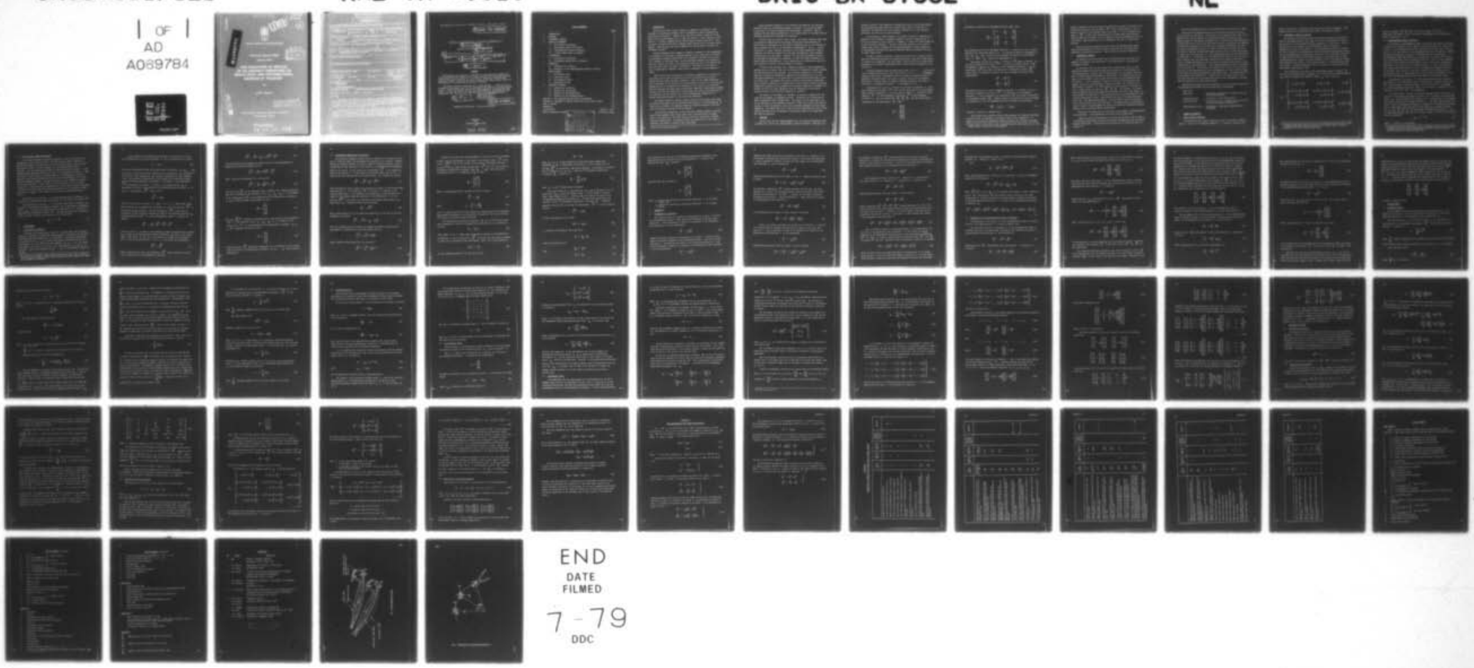

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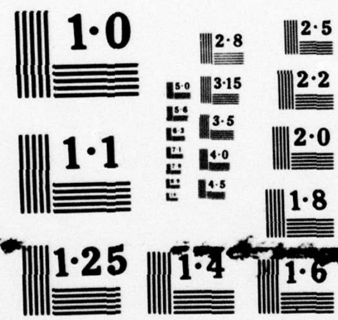

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THE EQUATIONS OF MOTION OF AN AIRCRAFT EMBRACING ITS WHOLE-BODY AND DEFORMATIONAL DEGREES OF FREEDOM

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THE EQUATIONS OF MOTION OF AN AIRCRAFT EMBRACING ITS WHOLE-BODY AND DEFORMATIONAL DEGREES OF FREEDOM.

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SUMMARY

This Report is intended as a contribution to the problem of unifying and relating the approaches adopted by the flight dynamicist who makes small perturbation studies of the behaviour of an aircraft in, primarily, its whole-body degrees of freedom, and by the structural dynamicist who makes similar studies of the behaviour in its deformational degrees of freedom.

The Report outlines a framework for a common approach. It goes as far as deriving the equations of motion, and showing how the terms etc in these equations relate to those traditionally used by the flight dynamicist and structural dynamicist. It does not discuss the subsequent approximations, simplifications and assumptions that then have to be made to assist in solving the equations.

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1 INTRODUCTION

There has ever been a need to achieve a commonality in approach to the treatment of aeronautical problems, and to standardize as much as possible the nomenclature and notation. Indeed, one of the first reports to come out of the Royal Aircraft Establishment after it came into being on 1 April 1918 was on this subject[1]. A further landmark was the classic paper by Bryant and Gates[2]. With the advent of missiles the need to embrace their features led to the great work by Hopkin[3]. There has been the need to embrace also the aeroelastic features of deformable aircraft[4].

Recently Woodcock[5] has attempted to provide a framework of the fundamentals of the subject, but his paper, although rigorous, does to some extent fall short of providing the dynamics practitioner with a document from which he can work. J.C.A. Baldock, on the other hand, in an unpublished document, has attempted to meet more closely the needs of the dynamics practitioner, but, as he himself has said, his paper lacks the rigour of Woodcock's. This paper is an attempt to meet the same needs as Baldock's paper, whilst attempting to retain Woodcock's rigour.

One of the current difficulties in achieving a commonality in approach is that there are two types of dynamicist, who are concerned with the overall problem from two widely different viewpoints. There is the flight dynamicist, who is interested in the aircraft as a flying machine, and there is the structural dynamicist, who is interested in the behaviour of the aircraft as a structure. Both are concerned with whole-body perturbations (shunting*, sideslipping, heaving, rolling, pitching and yawing) from a datum flight state and with deformations of the structure, but they choose to set up the mathematical formulation of a problem differently because of their different viewpoints.

The flight dynamicist imagines himself seated in the aircraft under study. He observes particularly how the aircraft moves relative to the datum motion, as well as how it deforms. He therefore takes his frame of reference in the aircraft, and chooses a system of axes accordingly.

The structural dynamicist, on the other hand, imagines himself in an aircraft flying alongside the aircraft under study, and moving with its datum motion. He observes how the study aircraft moves and, particularly, deforms from this vantage point. He therefore takes the datum motion as his frame of reference, and chooses a system of axes accordingly.

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* That is, fore-and-aft motion.

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Being interested primarily in the structural deformations the structural dynamicist takes the deformational degrees of freedom - the structural modes - as the basis of a system of generalised displacement co-ordinates. When he considers the whole-body perturbations it is a simple matter to introduce them as additional generalised displacement co-ordinates.

With the frame of reference that he takes, the flight dynamicist finds that he can achieve simplifications by working with the whole-body perturbation velocities as co-ordinates, for they lead to simplifications in the equations of motion compared with the use of perturbation displacements. The perturbation velocities, of course, describe the motion of the flight dynamicist's frame of reference, but when he considers the structural deformations, these represent motions relative to this frame. Moreover, the structural deformations have to be described by displacement co-ordinates, so, when the flight dynamicist considers structural deformations, in his formulation of the equations of motion he has a mixture of velocity and displacement co-ordinates.

It will be seen therefore that, adequate although the flight dynamicist's approach may be as long as there are no structural deformations, making allowance for them presents him with problems. The structural dynamicist on the other hand has no problem in extending his approach from consideration of just the structural deformations to the embracing of the whole-body perturbations as well, as we have seen. Accordingly the overall problem of whole-body perturbations plus structural deformations is probably most easily studied using the structural dynamicist's approach, and accordingly the structural dynamicist is probably in a position to help the flight dynamicist with the formulation of the overall problem using his, the structural dynamicist's, approach.

The purpose of this Report is, therefore, to provide a framework for the study of the dynamics of deformable aircraft. Attention will be restricted to small perturbations from a datum motion. The subject will be developed as far as the derivation of the equations of motion, in which the various terms will be related to those traditionally used by the structural dynamicist and the flight dynamicist. The subject will not be taken as far as discussing the values of the various coefficients in the equations in particular cases, or the terms that may often be neglected.

2 NOTATION

We have seen that the flight dynamicist and the structural dynamicist view a problem each from his own, and different, frame of reference. They also use

different notation. More important, although they tend to use the same symbols for representing kindred quantities, because their frames of reference are different, the quantities that these symbols represent are in fact different quantities, and this is of course confusing.

As the purpose of this document is to attempt to achieve a commonality in approach to the treatment of dynamical problems it would be inappropriate to write this document in the notation or 'language' of either the flight dynamicist or the structural dynamicist. Instead, it has been decided to use a kind of lingua franca, which, although not the working language of either dynamicist, is nevertheless one which each should understand, much as biologists use latin as their lingua franca.

We shall, therefore, develop a system of notation that is logical, and in which the information necessary for understanding what a particular symbol means is carried in its symbolism. We shall adopt a notation that has in effect been proposed by R.H. Merson in unpublished work, but was reported by Hopkin[3] in his section M.1.

We shall represent points by upper case letters O, D, L, ..., and sets of three orthogonal directions by lower case letters o, d, ℓ, ... A fundamental concept in the subject of dynamics is that of a frame of reference, which is defined by the specification of a point, or origin, and a set of orthogonal directions. We shall denote the frame of reference defined by the origin O and the set of orthogonal directions o , for example, by Oo .

The position of a point P relative to the frame of reference Lℓ , say, may be denoted by the vector $x^{PL\ell}$, but as this vector does not in fact depend upon the set of directions ℓ , but only upon the origin L , we write it more simply as x^{PL} . If the resolutes of the vector x^{PL} in the orthogonal directions d , say, are respectively $x_{d_1}^{PL}$, $x_{d_2}^{PL}$, $x_{d_3}^{PL}$, then the vector may be represented by the column matrix x_d^{PL} , where

$$x_d^{PL} \equiv \begin{bmatrix} x_{d_1}^{PL} \\ x_{d_2}^{PL} \\ x_{d_3}^{PL} \end{bmatrix} , \tag{1}$$

with which is associated an antisymmetric matrix X_{dd}^{PL}, where

$$X_{dd}^{PL} \equiv \begin{bmatrix} 0 & -x_{d_3}^{PL} & x_{d_2}^{PL} \\ x_{d_3}^{PL} & 0 & -x_{d_1}^{PL} \\ -x_{d_2}^{PL} & x_{d_1}^{PL} & 0 \end{bmatrix} . \tag{2}$$

The orientation of a set of orthogonal direction ℓ relative to a set d may be denoted by the orientation matrix $S_{\ell d}$. An orientation matrix is orthogonal, so that $S_{\ell d}^{-1} = S_{d\ell}$, and, if both sets of orthogonal directions are right-handed, $|S_{\ell d}| = 1$, where $|S_{\ell d}|$ denotes the determinant of $S_{\ell d}$.

The lineal* velocity of a point D relative to the frame of reference Oo, say, may be denoted by the vector u^{DOo}. As with the position vector, it may be expressed in terms of its resolutes in the orthogonal directions d by the column matrix u_d^{DOo}, with which is associated an antisymmetric matrix U_{dd}^{DOo}. Lineal velocity is defined in terms of the rate of change of position by the relationships

$$\left.\begin{aligned} u_o^{DOo} &\equiv \dot{x}_o^{DO} \\ U_{oo}^{DOo} &\equiv \dot{X}_{oo}^{DO} \end{aligned}\right\} . \tag{3}$$

The angular velocity of a set of orthogonal directions d relative to a set o, say, may be denoted by the vector p^{do}. Its resolutes in the orthogonal directions ℓ may be denoted by the column matrix p_ℓ^{do}, with which is associated an antisymmetric matrix $P_{\ell\ell}^{do}$. Angular velocity is related to the rate of change of orientation of a set of orthogonal directions through the relationships

$$P_{dd}^{do} = S_{do}\dot{S}_{od} = -\dot{S}_{do}S_{od} \tag{4}$$

(see Ref 6, article 8.5, equation (4)).

The notation just proposed is heavy with dressings, but these are necessary in order to make the meaning of a particular symbol quite unambiguous. Of course, if in a specific problem all lineal velocities are relative to the same

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* We use *lineal* to mean *along a line*, as against *angular* which means *about a line*, keeping *linear* to mean *of the first degree*.

frame of reference, and if all vectors are expressed in terms of the same set of orthogonal directions, then the symbol u_d^{DOo} could be abbreviated to just u^D. However, and this is the important point, the symbol u^D can be expanded back again to u_d^{DOo} to make clear precisely what the symbol means should another frame of reference or another set of orthogonal directions subsequently arise in the problem. Many notations are not so expandable, and this is considered a weakness.

There are many laws representing well-known relationships which appear obvious when expressed in the full notation, which are not so apparent when expressed in abbreviated notations. Some of these are listed in Appendix A.

3 FUNDAMENTAL CONCEPTS

We postulate that the aircraft has an *undeformed shape*, which may, for example, be defined as the shape given in the manufacturer's 3-view general arrangement drawing.

We postulate also that the aircraft has a *datum motion*, and it is the dynamics of departures from this datum motion which are to be the subject of this investigation. We shall take the datum motion to be as general as we can make it without making the problem too complicated. The datum motion that we shall in fact consider is one in which the resolutes in the datum-motion axes directions d (to be introduced in section 4.2) of the lineal velocity and angular velocity of the datum-motion frame of reference Dd relative to the earth frame of reference Oo (these concepts to be introduced in sections 4.2 and 4.1 respectively), as given by u_d^{DOo} and p_d^{do}, are constant. Of course, some problems may require more general datum motions than this, and indeed, in some problems, the main part of the problem may be the very determination of the so-called datum motion itself. This would be the case if the so-called datum motion were some complicated manoeuvre. But this would be another problem in itself, which we shall not enter into here. As already stated, we are going to limit our subject of investigation to that of the dynamics of departures from an assumed, albeit relatively simple, datum motion.

Because of its elasticity the structure of the aircraft is deformed during the datum motion. Such deformations are called *static deformations*.

Departures from the datum motion are called *perturbations*, and are made up of whole-body *translations*, *rotations* and further deformations which are called *dynamic deformations*. Translations and rotations will be defined precisely in
010 section 4.3.

The static deformations and all the perturbations are assumed to be small.

There are five identifiable states of the aircraft which we shall have reason to refer to (see Fig 1). First there is the *perturbed state:* this is the general state, and the absence of a specific subscript will indicate that a quantity refers to this general perturbed state. Second there is the *datum-motion or equilibrium state:* this is the state when there are no perturbations, implying no whole-body translations or rotations and no dynamic deformations, but there are static deformations, and also, of course, any control settings, such as of the throttle, aileron, elevator and rudder, must be such as to trim the aircraft in speed, roll, pitch and yaw, etc. This equilibrium state will be indicated by the subscript $\mathcal{E}$. Third there is the *undeformed-body state:* this is the state when, although there are whole-body translations and rotations, there are no static or dynamic deformations, but the control settings are the same as in the datum-motion or equilibrium state. This undeformed-body state will be indicated by the subscript $\mathcal{B}$. Fourth there is the *datum state:* this is the state when there are no perturbations and no static deformations, but the control settings are the same as in the datum-motion or equilibrium state. This datum state will be indicated by the subscript $\mathcal{D}$. Finally there is the *base state:* this is the state when there are no perturbations and no static deformations, and moreover all controls are at their zero settings.

Having defined these five states precisely, let us now relist them in the following way which exhibits better perhaps their relationships.

base state		
datum state	=	base state + controls set to provide trim for the equilibrium state
equilibrium state	=	datum state + static deformations
perturbed state	=	equilibrium state + whole-body translations and rotations and dynamic deformations
undeformed-body state	=	datum state + whole-body translations and rotations.

4 FRAMES OF REFERENCE

4.1 Earth frame of reference

We introduce an *earth frame of reference* fixed in the earth, having an *origin* O and a *set of orthogonal directions* o oriented in some way with

respect to the earth, but with the third direction vertically downwards. This system of axes* is an 'earth-fixed axes system', as defined by Hopkin[3].

4.2 Datum-motion frame of reference

We introduce a *datum-motion frame of reference* which may be considered as fixed in the datum motion, so that, when there are no perturbations - that is, the aircraft is in the equilibrium state, datum state, or base state - then of course it is at rest relative to the datum-motion frame of reference. Specifically, we take as the *origin* D of our datum-motion frame of reference the point occupied, in the datum state, by some suitably-chosen reference point, such as the manufacturer's datum point, and as the *set of orthogonal directions* d some axes which coincide with the directions, in the datum state, of some suitably-chosen axes, such as the axes determined by the 3-view general arrangement drawing of the aircraft. This system of axes* is a 'datum-attitude earth axes system', as defined by Hopkin[3], and is the same as the 'constant-velocity axes system' which Woodcock[5] uses.

The orientation of the axes of the datum-motion frame of reference Dd relative to the earth frame of reference Oo is given by the orientation matrix S_{do}, which is equivalent to Hopkin's[3] matrix S_{ϕ_e}, and is given by

$$S_{do} = \begin{bmatrix} \cos\phi_2^{do}\cos\phi_3^{do} & \cos\phi_2^{do}\sin\phi_3^{do} & -\sin\phi_2^{do} \\ \begin{array}{c}\sin\phi_1^{do}\sin\phi_2^{do}\cos\phi_3^{do} \\ -\cos\phi_1^{do}\sin\phi_3^{do}\end{array} & \begin{array}{c}\sin\phi_1^{do}\sin\phi_2^{do}\sin\phi_3^{do} \\ +\cos\phi_1^{do}\cos\phi_3^{do}\end{array} & \sin\phi_1^{do}\cos\phi_2^{do} \\ \begin{array}{c}\cos\phi_1^{do}\sin\phi_2^{do}\cos\phi_3^{do} \\ +\sin\phi_1^{do}\sin\phi_3^{do}\end{array} & \begin{array}{c}\cos\phi_1^{do}\sin\phi_2^{do}\sin\phi_3^{do} \\ -\sin\phi_1^{do}\cos\phi_3^{do}\end{array} & \cos\phi_1^{do}\cos\phi_2^{do} \end{bmatrix} \quad \text{......} \ (5)$$

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* A *system of axes* consists of both a set of orthogonal directions and an origin. It therefore defines a frame of reference, but clearly within a frame of reference there can be many systems of axes.

where the elements ϕ_1^{do}, ϕ_2^{do}, ϕ_3^{do} of the column matrix ϕ^{do} denote respectively the angles of bank, inclination and heading, and are equivalent to Hopkin's quantities Φ, Θ, Ψ .

4.3 Undeformed-body frame of reference

We introduce an *undeformed-body frame of reference* which may be considered as fixed in the aircraft, so that, when there are no deformations - that is, the aircraft is in the undeformed-body state, datum state, or base state - then of course it is at rest relative to the undeformed-body frame of reference. Depending upon the precise way in which it is defined there are an infinity of undeformed-body frames of reference, but specifically, we take as the *origin* B of our undeformed-body frame of reference the point occupied, in the undeformed-body state, by the suitably-chosen reference point, such as the manufacturer's datum point, and as the *set of orthogonal directions* b some suitably-chosen axes, such as the axes determined by the 3-view general arrangement drawing of the aircraft. This system of axes* may be considered as an adaptation to a deforming aircraft of a 'body-fixed axes system' as defined by Hopkin[3], and as akin to the 'no-deformation-body-fixed axes system' which Woodcock[5] uses.

The position of the origin B of the undeformed-body frame of reference Bb relative to the datum-motion frame of reference Dd is given by the vector x^{BD} , which represents the translations of the aircraft when it is perturbed. In its column matrix representation x_d^{BD} it is equivalent to Woodcock's[5] set of quantities $x_1^{(c)}$, $y_1^{(c)}$, $z_1^{(c)}$ (see Appendix B).

The orientation of the axes of the undeformed-body frame of reference Bb relative to the datum-motion frame of reference Dd is given by the orientation matrix S_{bd} , which represents the rotations of the aircraft when it is perturbed, and which is equivalent to Woodcock's[5] matrix S . For small perturbations the order in which the rotations are applied is commutative. We denote the rotations by the column matrix ϕ^{bd} with which is associated the antisymmetric matrix Φ^{bd} , so that ϕ_1^{bd}, ϕ_2^{bd}, ϕ_3^{bd} are equivalent to Woodcock's[5] quantities ϕ, θ, ψ . For small perturbations the orientation matrix is given in terms of the rotations by the relationship

$$S_{bd} \simeq I - \Phi^{bd} \qquad (6)$$

where I is the 3 × 3 unit matrix.

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* A *system of axes* consists of both a set of orthogonal directions and an origin. It therefore defines a frame of reference, but clearly within a frame of reference there can be many systems of axes.

4.4 Discrete-lump frames of reference

It is often convenient for analysis purposes to treat an aircraft as an assemblage of rigid 'discrete lumps'. For each discrete lump we introduce a *discrete-lump frame of reference*, fixed in the discrete lump, so that of course each discrete lump is at rest relative to its own discrete-lump frame of reference. For each discrete-lump frame of reference we take as *origin* L some suitably-chosen reference point, and as the *set of orthogonal directions* ℓ some axes which we shall choose so that, when there are no deformations, they are parallel to the suitably-chosen axes such as the axes determined by the 3-view general arrangement drawing. The axes directions ℓ are, therefore, parallel to the axes directions d when there are no perturbations, and to the axes directions b when there are no deformations. In some cases it may be convenient to choose the discrete-lump axes directions differently: this presents no basic problem, but it makes it less simple to relate them to the axes directions d and b .

The position of the origin L of a discrete-lump frame of reference $L\ell$ relative to the datum-motion frame of reference Dd is given by the vector x^{LD} .

The orientation of the axes ℓ of a discrete-lump frame of reference $L\ell$ relative to the datum-motion frame of reference Dd is given by the orientation matrix $S_{\ell d}$. We denote the rotations by the column matrix $\phi^{\ell d}$ with which is associated the antisymmetric matrix $\Phi^{\ell d}$. For small rotations the orientation matrix is given in terms of the rotations by the relationship

$$S_{\ell d} \simeq I - \Phi^{\ell d} \quad . \tag{7}$$

5 CO-ORDINATES

5.1 Generalised co-ordinates

As stated in the Introduction, section 1, we shall adopt the structural dynamicist's approach, and take the various degrees of freedom of the aircraft - both whole-body and deformational - as the basis of a system of generalised co-ordinates* $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_N$, which we may denote collectively by the $N \times 1$ column matrix $\bar{q}$. We shall use the co-ordinates $\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_6$ for *whole-body degrees of freedom* - $\bar{q}_1, \bar{q}_2, \bar{q}_3$, for translations and $\bar{q}_4, \bar{q}_5, \bar{q}_6$ for rotations - and the co-ordinates $\bar{q}_7, \bar{q}_8, \ldots, \bar{q}_N$ for *deformational degrees of freedom*.

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* We introduce the somewhat clumsy symbols with bars at this stage in order to keep the symbols without bars for the incremental quantities which will appear more frequently subsequently.

We shall express the generalised co-ordinates $\bar{q}$ as the sum of a value in the datum motion $q_{\mathcal{E}}$ plus an increment q due to perturbations, so we have

$$\bar{q} = q_{\mathcal{E}} + q \quad . \tag{8}$$

We take the datum-motion values of the whole-body co-ordinates $q_{1_{\mathcal{E}}}, q_{2_{\mathcal{E}}}, \ldots, q_{6_{\mathcal{E}}}$ to be zero, so that the first six elements of the column matrix $q_{\mathcal{E}}$ in equation (8) are zero. The column matrix $q_{\mathcal{E}}$ therefore represents the static deformations. The position of any particle of the aircraft may be expressed directly in terms of these generalised co-ordinates. For example, if the displacement $x^{PP_{\mathcal{D}}}$ of a particle at a point P relative to its datum position $P_{\mathcal{D}}$ is represented in terms of its resolutes in the datum-motion axes directions d by the 3×1 column matrix $x_d^{PP_{\mathcal{D}}}$, then we write

$$x_d^{PP_{\mathcal{D}}} = \mathscr{F}(\bar{q}) \tag{9}$$

(see Ref 6, article 8.9, equation (1)), where $\mathscr{F}$ is a 3×1 modal shape column matrix function of the generalised co-ordinates $\bar{q}$ and of the position $x^{P_{\mathcal{D}}D}$ of the particle being considered. Now the displacement $x_d^{PP_{\mathcal{D}}}$ of the particle represented in this way may be expressed in terms of its undeformed position $P_{\mathcal{B}}$, the undeformed-body origin B, and the datum-motion origin D through the relation

$$x_d^{PP_{\mathcal{D}}} = x_d^{BD} + x_d^{P_{\mathcal{B}}B} + x_d^{PP_{\mathcal{B}}} - x_d^{P_{\mathcal{D}}D} \quad . \tag{10}$$

Since the point $P_{\mathcal{D}}$ is fixed in the datum-motion frame of reference and the point $P_{\mathcal{B}}$ is fixed in the undeformed-body frame of reference, and since, in the datum state, these points $P_{\mathcal{D}}$ and $P_{\mathcal{B}}$ are coincident, as indeed are the origins D and B, and the axes directions d and b, then we have the relationship

$$x_d^{P_{\mathcal{D}}D} = x_b^{P_{\mathcal{B}}B} \quad . \tag{11}$$

Hence, using equation (A-3), and eliminating $x_b^{P_{\mathcal{B}}B}$ between equations (10) and (11), we obtain for the displacement of a particle

$$x_d^{PP_{\mathscr{D}}} = x_d^{BD} + (S_{db} - I)x_d^{P_{\mathscr{D}}D} + x_d^{PP_{\mathscr{B}}} \quad . \tag{12}$$

Since the rotations are assumed to be small, we may use the approximation of equation (6), and write equation (12) as

$$x_d^{PP_{\mathscr{D}}} \simeq x_d^{BD} + \phi^{bd} x_d^{P_{\mathscr{D}}D} + x_d^{PP_{\mathscr{B}}} \tag{13}$$

which, using the relationship (A-1), we may write

$$x_d^{PP_{\mathscr{D}}} \simeq x_d^{BD} - X_{dd}^{P_{\mathscr{D}}D} \phi^{bd} + x_d^{PP_{\mathscr{B}}} \quad . \tag{14}$$

The first term x_d^{BD} on the right-hand side of equation (14) represents displacement due to translation, and therefore, since $q_{1_{\mathscr{E}}} = q_{2_{\mathscr{E}}} = q_{3_{\mathscr{E}}} = 0$, the elements of the column matrix x_d^{BD} may be identified with the generalised co-ordinates q_1, q_2, q_3 which represent translations, so that

$$x_d^{BD} \equiv \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad . \tag{15}$$

The term $X_{dd}^{P_{\mathscr{D}}D} \phi^{bd}$ represents displacement due to rotation about the datum-motion origin D , and therefore, since $q_{4_{\mathscr{E}}} = q_{5_{\mathscr{E}}} = q_{6_{\mathscr{E}}} = 0$, the elements of the column matrix ϕ^{bd} may be identified with the generalised co-ordinates q_4, q_5, q_6 which represent rotations, so that

$$\phi^{bd} \equiv \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad . \tag{16}$$

Finally the term $x_d^{PP_{\mathscr{B}}}$ represents displacement due to deformation, and is therefore a function of the generalised co-ordinates $\bar{q}_7, \bar{q}_8, \ldots, \bar{q}_N$ which represent deformations.

5.2 Intermediate generalised co-ordinates

The structural dynamicist usually finds it convenient to introduce the concept of 'discrete lumps' for determining the properties of an aircraft. However, he may use different families of discrete lumps for different purposes, such as determining the mass properties of the aircraft, or determining the aerodynamics. Whatever the purpose, he will express the displacement $x_d^{PP_{\mathscr{D}}}$ of the particle in terms of the position L of the origin of the discrete lump, say the h^{th}, to which the particle belongs and of its datum position $L_{\mathscr{D}}$, through the relation

$$x_d^{PP_{\mathscr{D}}} = x_d^{LL_{\mathscr{D}}} + x_d^{PL} - x_d^{P_{\mathscr{D}}L_{\mathscr{D}}} \quad . \tag{17}$$

Since the discrete lump is rigid, the resolutes with respect to the discrete-lump axes directions ℓ of the position x_ℓ^{PL} of the point P relative to the discrete-lump origin L are constant, and thereby equal to the resolutes with respect to the discrete-lump axes directions in the datum state d of the position $x_d^{P_{\mathscr{D}}L_{\mathscr{D}}}$ of the point $P_{\mathscr{D}}$ relative to the discrete-lump origin in the datum state $L_{\mathscr{D}}$. Therefore we have the relationship

$$x_\ell^{PL} = x_d^{P_{\mathscr{D}}L_{\mathscr{D}}} \quad . \tag{18}$$

Hence, using equation (A-3), we obtain from equations (17) and (18) for the displacement of a particle

$$x_d^{PP_{\mathscr{D}}} = x_d^{LL_{\mathscr{D}}} + (S_{d\ell} - I)x_\ell^{PL} \quad . \tag{19}$$

Since the deformations and rotations are assumed to be small, we may use the approximation of equation (7), and write equation (19) as

$$x_d^{PP_{\mathscr{D}}} \simeq x_d^{LL_{\mathscr{D}}} + \phi^{\ell d} x_\ell^{PL} \tag{20}$$

which, using the relationship (A-1), we may write

$$x_d^{PP_{\mathscr{D}}} \simeq x_d^{LL_{\mathscr{D}}} - X_{\ell\ell}^{PL}\phi^{\ell d} \quad . \tag{21}$$

Equation (21) shows that the displacement of the particle may be considered as made up of the displacement of the origin of the discrete lump, $x_d^{LL_{\mathscr{D}}}$, plus a term, $-X_{\ell\ell}^{PL}\phi^{\ell d}$, due to the rotation of the discrete lump about its origin L . For the h^{th} discrete lump we take the quantities $x_d^{LL_{\mathscr{D}}}$ and $\phi^{\ell d}$ as a sextet of intermediate generalised co-ordinates $\bar{q}_{h1}^*, \bar{q}_{h2}^*, \ldots, \bar{q}_{h6}^*$, which we denote collectively by the 6 × 1 column matrix $\bar{q}_h^*$, so that

$$\bar{q}_h^* \equiv \begin{bmatrix} x_d^{LL_{\mathscr{D}}} \\ \hline \phi^{\ell d} \end{bmatrix} . \tag{22}$$

Hence, by using equation (22), *equation* (21) may be written

$$x_d^{PP_{\mathscr{D}}} \mathrel{\hat{=}} J_h \bar{q}_h^* \tag{23}$$

where
$$J_h = \left[I \;\middle|\; -X_{\ell\ell}^{PL} \right] \tag{24}$$

is a 3 × 6 matrix coefficient whose elements are independent of the intermediate generalised co-ordinates, but do depend upon the position x_ℓ^{PL} within the discrete lump of the particle being considered.

The intermediate generalised co-ordinates $\bar{q}_h^*$ for the h^{th} discrete lump may be expressed in terms of the generalised co-ordinates $\bar{q}$ by a relation of the form

$$\bar{q}_h^* = \mathcal{H}_h(\bar{q}) \tag{25}$$

where $\mathcal{H}_h(\bar{q})$ is a 6 × 1 modal shape column matrix function of the generalised co-ordinates $\bar{q}$ and of the position $x_d^{L_{\mathscr{D}}D}$ of the discrete lump being considered. For small static deformations and perturbations $\mathcal{H}_h(\bar{q})$ may be approximated by

$$\mathcal{H}_h(\bar{q}) \mathrel{\hat{=}} H_h \bar{q} \tag{26}$$

so that, combining equations (25) and (26), we have

$$\bar{q}^*_h \triangleq H_h \bar{q} \tag{27}$$

where H_h is a $6 \times N$ modal shape matrix coefficient whose elements are independent of the N generalised co-ordinates $\bar{q}$, but do depend upon the position $x_d^{L\mathscr{D}}$ of the discrete lump being considered. Accordingly if we denote the ijth element of H_h by H_{hij}, then the ith intermediate generalised co-ordinate $\bar{q}^*_{hi}$ is given by

$$\bar{q}^*_{hi} \triangleq \sum_{j=1}^{N} H_{hij} \bar{q}_j \tag{28}$$

where $\bar{q}_j$ is the jth generalised co-ordinate.

For a given value of i the quantities H_{hij} may be written as an $n \times N$ matrix with H_{hij} as the hjth element, where n is the number of discrete lumps. Each column of this $n \times N$ matrix gives the shape of the mode for the corresponding generalised co-ordinate $\bar{q}_j$, the shape being expressed as the variation of the ith intermediate generalised co-ordinate $\bar{q}^*_{hi}$. Combining equations (23) and (27) we see that the displacement $x_d^{PP\mathscr{D}}$ of a particle may be written

$$x_d^{PP\mathscr{D}} \triangleq J_h H_h \bar{q} \tag{29}$$

so that, from equation (9), we have

$$\mathscr{F}(\bar{q}) \triangleq J_h H_h \bar{q} \quad . \tag{30}$$

In conformity with equation (8) we may write

$$\bar{q}^*_h = q^*_{h_g} + q^*_h \tag{31}$$

where, from equation (27),

$$q^*_{h_g} \triangleq H_h q_g \tag{32}$$

and

$$q^*_h \triangleq H_h q \tag{33}$$

are respectively the value of the intermediate generalised co-ordinate in the datum motion and the increment due to the perturbations. It follows from equation (22) that $q^*_{h_{\mathscr{E}}}$ is given by

$$q^*_{h_{\mathscr{E}}} = \begin{bmatrix} x_d^{L_{\mathscr{E}}L_{\mathscr{D}}} \\ ----- \\ \phi^{\ell_{\mathscr{E}}d} \end{bmatrix} \tag{34}$$

and hence that q^*_h is given by

$$q^*_h = \begin{bmatrix} x_d^{LL_{\mathscr{E}}} \\ ---- \\ \phi^{\ell\ell_{\mathscr{E}}} \end{bmatrix} \tag{35}$$

where $L_{\mathscr{E}}$ denotes the position of the discrete-lump origin L in the datum motion, and

$\ell_{\mathscr{E}}$ denotes the direction of the discrete-lump axes ℓ in the datum motion.

6 KINEMATICS

6.1 *Kinematics of a particle*

The lineal position of a particle of the aircraft at a point P relative to the datum-motion origin D when resolved with respect to the earth axes directions o may be expressed in terms of its resolutes in the datum-motion axes directions d by the relationship

$$x_o^{PD} = S_{od}x_d^{PD} \quad . \tag{36}$$

Likewise the lineal velocity of the particle at the point P relative to the frame of reference determined by the datum-motion origin D and the earth axes directions o and resolved with respect to the earth axes directions o may also be expressed in terms of its resolutes in the datum-motion axes directions d by the relationship

$$u_o^{PDo} = S_{od}u_d^{PDo} \quad . \tag{37}$$

Similarly the lineal velocity of the particle at the point P relative to the earth frame of reference and resolved with respect to the earth axes directions o may be expressed in terms of its resolutes with respect to the datum-motion axes directions d by the relationship

$$u_o^{POo} = S_{od} u_d^{POo} \quad . \tag{38}$$

Differentiating equation (36) with respect to time t , which we denote by a dot, we obtain

$$u_o^{PDo} \equiv \dot{x}_o^{PD} = S_{od} \dot{x}_d^{PD} + \dot{S}_{od} x_d^{PD} \tag{39}$$

and therefore, eliminating u_o^{PDo} between equations (37) and (39), and using equation (4), we obtain the following relation between the lineal velocity and position of the particle at the point P relative to the frame of reference determined by the origin D and axes directions o when resolved with respect to the datum-motion axes directions d

$$u_d^{PDo} = \dot{x}_d^{PD} + P_{dd}^{do} x_d^{PD} \quad . \tag{40}$$

On differentiating with respect to time, equation (40) yields

$$\dot{u}_d^{PDo} = \ddot{x}_d^{PD} + P_{dd}^{do} \dot{x}_d^{PD} + \dot{P}_{dd}^{do} x_d^{PD} \quad . \tag{41}$$

Finally, the lineal acceleration of the particle at the point P relative to the earth frame of reference and resolved with respect to the earth axes directions o may be expressed in terms of its resolutes with respect to the datum-motion axes directions d by the relationship

$$f_o^{POo} = S_{od} f_d^{POo} \quad . \tag{42}$$

Differentiating equation (38) with respect to time, we obtain

$$f_o^{POo} \equiv \dot{u}_o^{POo} = S_{od} \dot{u}_d^{POo} + \dot{S}_{od} u_d^{POo} \tag{43}$$

and therefore, eliminating f_o^{POo} between equations (42) and (43), and using equation (4), we obtain the following relation between the lineal acceleration and velocity of the particle at the point P relative to the earth frame of reference when resolved with respect to the datum-motion axes directions d

$$f_d^{POo} = \dot{u}_d^{POo} + P_{dd}^{do} u_d^{POo} \quad . \tag{44}$$

We now express the lineal velocity of P relative to O as the sum of the lineal velocity of P relative to D , and that of D relative to O

$$u_d^{POo} = u_d^{DOo} + u_d^{PDo} \quad . \tag{45}$$

Differentiating equation (45) with respect to time we obtain

$$\dot{u}_d^{POo} = \dot{u}_d^{DOo} + \dot{u}_d^{PDo} \quad . \tag{46}$$

Hence, eliminating $\dot{u}_d^{POo}$, $\dot{u}_d^{PDo}$, u_d^{POo}, u_d^{PDo} between *equations* (44), (46), (41), (45) and (40) we obtain the following relation between the lineal acceleration of the particle at the point P relative to the earth frame of reference and the position of the particle relative to the origin D when resolved with respect to the datum-motion axes directions d

$$f_d^{POo} = \dot{u}_d^{DOo} + P_{dd}^{do} u_d^{DOo} + \ddot{x}_d^{PD} + 2P_{dd}^{do}\dot{x}_d^{PD} + {P_{dd}^{do}}^2 x_d^{PD} + \dot{P}_{dd}^{do} x_d^{PD} \quad . \tag{47}$$

Now, as stated in section 3, we postulated that, in the datum motion, the resolutes in the datum-motion axes directions d of the lineal and angular velocities of the datum-motion origin relative to the earth frame of reference, u_d^{DOo} and P_{dd}^{do} , are constant, and therefore the terms involving $\dot{u}_d^{DOo}$ and $\dot{P}_{dd}^{do}$ in equation (47) are zero. Therefore, with our postulates, equation (47) becomes

$$f_d^{POo} = P_{dd}^{do} u_d^{DOo} + \ddot{x}_d^{PD} + 2P_{dd}^{do}\dot{x}_d^{PD} + {P_{dd}^{do}}^2 x_d^{PD} \tag{48}$$

where the first term on the right-hand side is the acceleration of the datum-motion origin D relative to the earth frame of reference Oo , and the second,
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accelerations of the particle at point P relative to the datum-motion frame of reference Dd . (See Ref 7, section 12.3.)

Now

$$x_d^{PD} = x_d^{L\mathscr{D}D} + x_d^{P\mathscr{D}L\mathscr{D}} + x_d^{PP\mathscr{D}} \tag{49}$$

which, using equations (18), (23) and (31), we may write in terms of intermediate generalised co-ordinates as

$$x_d^{PD} \simeq x_d^{L\mathscr{D}D} + x_\ell^{PL} + J_h q^*_{h_{\mathscr{E}}} + J_h q^*_h \tag{50}$$

where $x_d^{L\mathscr{D}D}$, x_ℓ^{PL}, J_h and $q^*_{h_{\mathscr{E}}}$ are all constant with respect to time. Substituting for x_d^{PD} from equation (50) into equation (48) we obtain for the lineal accleeration of a particle relative to the earth frame of reference with respect to the datum-motion axes directions d in terms of the intermediate generalised co-ordinates q_h^*

$$f_d^{POo} \simeq P_{dd}^{do} u_d^{DOo} + P_{dd}^{do^2} x_d^{L\mathscr{D}D} + P_{dd}^{do^2} x_\ell^{PL} + P_{dd}^{do^2} J_h q^*_{h_{\mathscr{E}}} + J_h \ddot{q}_h^* + 2P_{dd}^{do} J_h \dot{q}_h^* + P_{dd}^{do^2} J_h q_h^* \; . \tag{51}$$

6.2 Kinematics of the undeformed-body frame of reference

The translation of the origin B of the undeformed-body frame of reference relative to its datum position D is given by equation (15).

Now the lineal velocity of the origin B of the undeformed-body frame of reference relative to the earth frame of reference when resolved with respect to the datum-motion axes directions d can be written as

$$u_d^{BOo} = u_d^{DOo} + u_d^{BDo} \; . \tag{52}$$

Substituting for u_d^{BDo} from equation (40) with the point P replaced by B we obtain

$$u_d^{BOo} = u_d^{DOo} + \dot{x}_d^{BD} + P_{dd}^{do} x_d^{BD} \tag{53}$$

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which, using equation (15), we may write in terms of the generalised co-ordinates q_1, q_2, q_3 and the associated generalised velocities $\dot{q}_1$, $\dot{q}_2$, $\dot{q}_3$ as

$$u_d^{BOo} = u_d^{DOo} + \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} . \tag{54}$$

The lineal velocity of the origin B of the undeformed-body frame of reference relative to the earth frame of reference when resolved now with respect to the undeformed-body axes directions b is given by

$$u_b^{BOo} = S_{bd} u_d^{BOo} . \tag{55}$$

Substituting for S_{bd} from equation (6) and for u_d^{BOo} from equation (54) we obtain, for small perturbations,

$$u_b^{BOo} \triangleq (I - \Phi^{bd}) \left(u_d^{DOo} + \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \right) \tag{56}$$

which, using equations (A-1) and (16) yields, to first order,

$$u_b^{BOo} \triangleq u_d^{DOo} + \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + U_{dd}^{DOo} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} . \tag{57}$$

In the datum motion the undeformed-body and the datum-motion frames of reference are coincident, and therefore u_d^{DOo} is the value of both u_d^{BOo} and u_b^{BOo} in the datum motion.

Now, equation (54) shows, therefore, that, when the angular velocity p_d^{do} of the datum-motion frame of reference relative to the earth frame of reference, and hence P_{dd}^{do}, is zero, which is the situation most commonly considered by the

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structural dynamicist, the generalised velocities $\dot{q}_1, \dot{q}_2, \dot{q}_3$ may be identified with the perturbations in the lineal velocity of the undeformed-body origin when resolved with respect to the datum-motion axes directions which the structural dynamicist uses, although they may not be so identified in general. Equation (57), however, shows that this identification is not true with the undeformed-body axes directions which the flight dynamicist uses, even when P_{dd}^{do} is zero, owing to the presence of the term which may alternatively be written as $U_{dd}^{DOo}\phi^{bd}$. The flight dynamicist usually writes the elements of the lineal velocity u_b^{BOo} as u, v, w, and their datum-motion values as given by u_d^{DOo}, as u_e, v_e, w_e. The differences between these two trios of lineal velocity he writes as u', v', w'. We have hereby established that these quantities are not equal to $\dot{q}_1, \dot{q}_2, \dot{q}_3$ respectively, but, by equation (57), they are given by

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \triangleq \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + U_{dd}^{DOo} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} . \tag{58}$$

The rotation of the axes directions b of the undeformed-body frame of reference relative to their datum directions d is given by equation (16).

Now the angular velocity of the axes directions b of the undeformed-body frame of reference relative to the earth frame of reference when resolved with respect to the datum-motion axes directions d can be written as

$$P_{dd}^{bo} = P_{dd}^{do} + P_{dd}^{bd} . \tag{59}$$

Substituting for P_{dd}^{bd} from equation (4) with the directions o replaced by b we obtain

$$P_{dd}^{bo} = P_{dd}^{do} - S_{db}\dot{S}_{bd} \tag{60}$$

which, using equation (6), we may write approximately as

$$P_{dd}^{bo} \triangleq P_{dd}^{do} + \dot{\phi}^{bd} . \tag{61}$$

Now, using equation (16), we may write equation (61) in terms of the generalised velocities $\dot{q}_4$, $\dot{q}_5$, $\dot{q}_6$ as

$$p_d^{bo} \simeq p_d^{do} + \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \quad . \tag{62}$$

The angular velocity of the axes directions b of the undeformed-body frame of reference relative to the earth frame of reference when resolved now with respect to the undeformed-body axes directions b is given by

$$p_b^{bo} = S_{bd} p_d^{bo} \quad . \tag{63}$$

Substituting for S_{bd} from equation (6) and for p_d^{bo} from equation (62) we obtain, for small perturbations,

$$p_b^{bo} \simeq (I - \phi^{bd}) \left(p_d^{do} + \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} \right) \tag{64}$$

which, using equations (A-1) and (16), yields, to first order,

$$p_b^{bo} \simeq p_d^{do} + \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} \quad . \tag{65}$$

In the datum motion the undeformed-body and the datum-motion frames of reference are coincident, and therefore p_d^{do} is the value of both p_d^{bo} and p_b^{bo} in the datum motion.

Equation (62) shows that the generalised velocities $\dot{q}_4$, $\dot{q}_5$, $\dot{q}_6$ may be identified with the perturbations in the angular velocity of the undeformed-body axes directions when resolved with respect to the datum-motion axes directions which the structural dynamicist uses. Equation (65), however, shows that this

identification is not true in general when the angular velocity is resolved with respect to the undeformed-body axes directions which the flight dynamicist uses, owing to the presence of the term which may alternatively be written as $P_{dd}^{do}\phi^{bd}$. The flight dynamicist usually writes the elements of the angular velocity p_b^{bo} as p, q, r , and their datum-motion values, as given by p_d^{do} as p_e, q_e, r_e . The differences between these two trios of angular velocity he writes as p', q', r' . We have hereby established that these quantities are not equal to $\dot{q}_4, \dot{q}_5, \dot{q}_6$ respectively, in general, but by equation (65), they are given by

$$\begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} \simeq \begin{bmatrix} \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} + P_{dd}^{do} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} \tag{66}$$

(see Ref 3, equation (5-13)).

7 APPLIED FORCES

7.1 Generalised forces

Let the applied forces on the particle of the aircraft at point P be denoted by the vector k , and let the resolutes of this vector in the datum-motion axes direction d be represented by the column matrix k_d .

The total work done Δ_A by the applied forces k_d on each particle of the aircraft in an infinitesimal displacement $\delta x_d^{PP_\mathscr{D}}$ of each particle from its position P is given by

$$\Delta_A = \sum_{PA} \tilde{k}_d \delta x_d^{PP_\mathscr{D}} \tag{67}$$

where $\sum_{PA}$ indicates summation over all particles of the aircraft, and the tilde above a matrix symbol denotes transposition, so that $\tilde{k}_d$ is a 1×3 row matrix.

Now, from equation (9),

$$\delta x_d^{PP_\mathscr{D}} = \frac{\partial \mathscr{F}}{\partial \bar{q}} \delta \bar{q} \tag{68}$$

where $\frac{\partial \mathscr{F}}{\partial \bar{q}}$ is a $3 \times N$ matrix.

Therefore, equation (67) may be written

$$\Delta_A = \tilde{z}\delta\bar{q} = \widetilde{\delta\bar{q}}\, z \tag{69}$$

where z is an $N \times 1$ column matrix of N generalised applied forces, and is given by

$$z = \sum_{PA} \frac{\partial\widetilde{\mathscr{F}}}{\partial\bar{q}} k_d \quad . \tag{70}$$

Now, from equation (30), we may write

$$\frac{\partial\widetilde{\mathscr{F}}}{\partial\bar{q}} \simeq (1 + O(\bar{q}))\tilde{H}_h\tilde{J}_h \quad . \tag{71}$$

We may also write

$$k_d \simeq k_{d_\mathscr{D}} + \frac{\partial k}{\partial\bar{q}}\bar{q} \tag{72}$$

where $k_{d_\mathscr{D}}$ is a column matrix of applied forces on the particle in the datum state, and

$\frac{\partial k_d}{\partial\bar{q}}$ is a $3 \times N$ matrix of applied force derivatives for the particle.

Therefore, equation (70) may be approximated by

$$z \simeq \sum_{PA} (1 + O(\bar{q}))\tilde{H}_h\tilde{J}_h\left(k_{d_\mathscr{D}} + \frac{\partial k_d}{\partial\bar{q}}\bar{q}\right) \quad . \tag{73}$$

Now, the summations in equation (73) arising from the $O(\bar{q})$ term will be small compared with the corresponding summations arising from the 1 term in the first factor, and may, therefore, on order arguments, legitimately be neglected. However, there is no guarantee that the summations arising from the $\frac{\partial k_d}{\partial\bar{q}}\bar{q}$ term in the last factor will be small compared with the summations arising from the $k_{d_\mathscr{D}}$ term. This is particularly so in regard to the aerodynamic forces. It stems from the fact that summations arising from the $k_{d_\mathscr{D}}$

term can be small, or even zero, in which case the summations arising from the $\frac{\partial k_d}{\partial \bar{q}} \bar{q}$ term are the dominant ones. For example, in the datum motion the aerodynamic pitching moment on an aircraft about its centre of mass is usually zero, so the pitching moment associated with the perturbations - corresponding to the $\frac{\partial k_d}{\partial \bar{q}} \bar{q}$ term - are clearly the dominant ones. We should, therefore, lose the essential perturbation forces if we neglected the summations arising from the $\frac{\partial k_d}{\partial \bar{q}} \bar{q}$ term and retained only the summations arising from the $k_{d\mathscr{D}}$ term. As we have said, similar arguments do not apply in regard to the summations arising from the $O(\bar{q})$ term in the first factor compared with those arising from the 1 term. It is important to make this observation, because the summations arising from the $O(\bar{q})$ term and from the $\frac{\partial k_d}{\partial \bar{q}} \bar{q}$ term are both formally of order $\bar{q}$, and therefore superficially one would not appear to be justified in retaining summations arising from one without retaining those arising from the other.

Accordingly, neglecting the summations arising from the $O(\bar{q})$ term in the first factor of equation (73), and rewriting the last factor, we obtain

$$z \bumpeq \sum_{PA} \tilde{H}_h \tilde{J}_h k_d \quad . \qquad (74)$$

We note that equation (74) is precisely what one would have obtained had one merely substituted for $\frac{\partial \mathscr{F}}{\partial \bar{q}}$ from equation (30) into equation (70). However, it is necessary to go through the analysis as above, for Woodcock[5] has shown that, on a strictly order of magnitude argument, equation (74) does not represent an adequate approximation. Nevertheless, although equation (74) is effectively the relationship that is used by structural dynamics practitioners, the justification for its use is certainly a matter that requires further investigation. (Equation (74) can, for example, be thought of as precisely what we require if we derive the equation of motion for the aircraft as $\sum_{PA} \left\{ \tilde{H}_h \tilde{J}_h \times \text{(equation of motion for a particle)} \right\}$, as we shall do in section 8.2.)

In like manner the total work done Δ_L by the applied forces k_d on each particle of a discrete lump in an infinitesimal displacement $\delta x_d^{PP_{\mathscr{D}}}$ of each particle from its position P is given by

$$\Delta_L = \sum_{PL} \tilde{k}_d \delta x_d^{PP_{\mathscr{D}}} \qquad (75)$$

where $\sum_{PL}$ indicates summation over all particles of the discrete lump.

Now, from equation (23)

$$\delta x_d^{PP_{\mathscr{D}}} \triangleq J_h \delta \bar{q}_h^* \quad . \qquad (76)$$

Therefore, equation (75) may be written

$$\Delta_L = \tilde{z}_h \delta \bar{q}_h^* = \delta \tilde{\bar{q}}_h^* z_h \qquad (77)$$

where z_h is a 6 × 1 column matrix of six intermediate generalised applied forces on the discrete lump. By going through analysis similar to that just gone through in regard to the generalised forces z , we can show that z_h is given approximately by

$$z_h \triangleq \sum_{PL} \tilde{J}_h k_d \quad . \qquad (78)$$

Eliminating k_d between equations (73) and (78), we see that the generalised forces z may be expressed in terms of the intermediate generalised forces z_h on the discrete lumps by the relation

$$z \triangleq \sum_{LA} \tilde{H}_h z_h \qquad (79)$$

where $\sum_{LA}$ indicates summation over all discrete lumps of the aircraft.

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7.2 Structural forces

The structural forces are produced by elastic strains in the structure, and possibly also from other causes such as mechanical friction, and are assumed to be determined entirely by the shape of the aircraft at any instant.

If the elastic strain energy W in the structure can be written in the form

$$W = \tfrac{1}{2}\tilde{\bar{q}} C_s \bar{q} \tag{80}$$

where C_s is an $N \times N$ symmetric matrix, then the generalised structural elastic forces are given by

$$-\frac{\partial W}{\partial \bar{q}} = -C_s \bar{q} \tag{81}$$

or, by virtue of equation (8), by

$$-\frac{\partial W}{\partial \bar{q}} = -C_s q_{\&} - C_s q \tag{82}$$

where the first term on the right-hand side represents the elastic forces associated with the static deformations, and the second term represents the elastic forces associated with the dynamic deformations.

If it is assumed that in addition there are generalised structural damping forces which may be represented by $B_s \dot{q}$, then the total generalised structural applied forces z_s are given by

$$z_s = z_{s_{\&}} - C_s q - B_s \dot{q} \tag{83}$$

where

$$z_{s_{\&}} = -C_s q_{\&} \tag{84}$$

are the generalised structural forces in the datum motion.

The elements of the structural stiffness matrix C_s and of the structural damping matrix B_s involving whole-body modes will, of course, be zero. Structural dynamicists usually denote the matrix C_s by E and the matrix B_s by D.

If the generalised co-ordinates are chosen to be normal co-ordinates, then the process which determines the modal shape matrices H_h also determines the modal frequencies $\omega_7, \omega_8, \ldots, \omega_N$ of the deformational modes. In this case, if we write the N × N diagonal matrix of modal frequencies as

$$\hat{\Omega} = \begin{bmatrix} 0 \ldots 0 & 0 & 0 \ldots 0 \\ \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ 0 \ldots 0 & 0 & 0 \ldots 0 \\ 0 \ldots 0 & \omega_7 & 0 \ldots 0 \\ 0 \ldots 0 & 0 & \omega_8 \ldots 0 \\ \ldots\ldots\ldots & \ldots & \ldots\ldots\ldots \\ 0 \ldots 0 & 0 & 0 \ldots \omega_N \end{bmatrix} \tag{85}$$

then the N × N structural stiffness matrix C_s will be diagonal, and given by

$$\hat{C}_s = \hat{\Omega}^2 \hat{A}_e \tag{86}$$

where A_e is the N × N *inertia matrix*, and the circumflexes are reminders that the matrices are here diagonal.

7.3 Gravitational forces

The orientation of the datum-motion frame of reference to the earth frame of reference is given by the orientation matrix S_{do}, given by equation (5).

Now, if g denotes the acceleration due to gravity, its resolutes in the orthogonal directions o are given by the column matrix

$$g_o = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \tag{87}$$

and hence its resolutes in the orthogonal directions d are given by the column matrix

$$g_d = S_{do} g_o = g S_{o_3 d} \tag{88}$$

where $S_{o_3 d}$ represents the column matrix given by

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$$S_{o_3d} = \begin{bmatrix} -\sin\phi_2^{do} \\ \sin\phi_1^{do}\cos\phi_2^{do} \\ \cos\phi_1^{do}\cos\phi_2^{do} \end{bmatrix} . \tag{89}$$

Therefore the gravitational force k_g on the particle of the aircraft at point P is given by

$$k_{g_d} = mg_d = mgS_{o_3d} \tag{90}$$

where m is the mass of the particle, so that, using equation (78), we have for the intermediate generalised gravitational forces z_{gh} on a discrete lump

$$z_{gh} \triangleq \left\{\sum_{PL} \tilde{J}_h m\right\} gS_{o_3d} . \tag{91}$$

Hence, using equation (79), we have for the generalised gravitational forces z_g on the aircraft

$$z_g \triangleq \left(\sum_{LA} \tilde{H}_h\left\{\sum_{PL} \tilde{J}_h m\right\}\right) gS_{o_3d} . \tag{92}$$

Equation (92) shows that, to the approximation that we have assumed, the generalised gravitational forces z_g do not depend upon the generalised co-ordinates $\bar{q}$, and they therefore remain equal to their value in the datum-motion z_{g_g} during the perturbations. Moreover, if the generalised gravitational forces in the datum motion are to remain constant, then the datum motion must be such that S_{o_3d} is constant: that is, that if the aircraft is turning, its angular velocity relative to the earth frame of reference must be about a vertical axis.

7.4 Aerodynamic forces

The assumptions that are usually made in the determination of the aerodynamic forces are that the atmosphere is still, that it is uniform, or at most stratified horizontally, and that the perturbation forces depend only upon the instantaneous values of the generalised co-ordinates q and of their derivatives

$\dot{q}$, as well as upon the frequencies of any oscillations, so that the generalised aerodynamic force z_a may be written

$$z_a \triangleq z_{a_{\mathcal{G}}} - C_a q - B_a \dot{q} \tag{93}$$

where $z_{a_{\mathcal{G}}}$ is the generalised aerodynamic force in the datum motion, and C_a, B_a are N × N aerodynamic stiffness and damping matrices respectively.

One usually normalizes the aerodynamic stiffness matrix C_a by the factor ρV^2 , where ρ is the ambient air density, and $V = |u_d^{DOo}|$ is the steady lineal speed of the aircraft as represented by the speed of the datum-motion origin D . C_a is therefore usually denoted by $\rho V^2 C$ by structural dynamicists, so that

$$\rho V^2 C \equiv C_a \quad . \tag{94}$$

Similarly the aerodynamic damping matrix B_a is usually normalised by the factor ρV , and therefore it is usually denoted by ρVB by structural dynamicists, so that

$$\rho VB \equiv B_a \quad . \tag{95}$$

Various methods can be employed to determine the aerodynamic forces, ranging from full lifting surface theories to various forms of strip theory. Let us consider how one would proceed on a strip theory approach, in which we shall make use of our division of the aircraft into discrete lumps.

We take our intermediate generalised co-ordinates $\bar{q}_h^*$ for the h^{th} discrete lump as given by equation (22), and we take the associated intermediate generalise generalised forces as given by equation (78). Then, assuming that the perturbations consist of oscillations of frequency ν , we write for the intermediate generalised aerodynamic force z_{ah}

$$\begin{aligned} z_{ah} \triangleq z_{ah_{\mathcal{D}}} &+ \frac{\partial z_{ah}}{\partial q_1^*}\bar{q}_1^* + \ldots + \frac{\partial z_{ah}}{\partial q_k^*}\bar{q}_k^* + \ldots + \frac{\partial z_{ah}}{\partial q_n^*}\bar{q}_n^* \\ &+ \frac{\partial z_{ah}}{\partial \dot{q}_1^*}\dot{q}_1^* + \ldots + \frac{\partial z_{ah}}{\partial \dot{q}_k^*}\dot{q}_k^* + \ldots + \frac{\partial z_{ah}}{\partial \dot{q}_n^*}\dot{q}_n^* \end{aligned} \tag{96}$$

where $\dfrac{\partial z_{ah}}{\partial q_l^*}$, $\dfrac{\partial z_{ah}}{\partial \dot{q}_l^*}$, etc are 6 × 6 matrices of aerodynamic derivatives appropriate to the frequency ν, and $z_{ah_{\mathscr{D}}}$ is the intermediate generalised aerodynamic force on the h^{th} discrete lump in the datum state. The inclusion of intermediate generalised co-ordinates other than the h^{th} allows for interference between discrete lumps, but often many of the possible interference terms will be ignored.

The aerodynamic derivatives will usually be expressed as the product of a non-dimensional form of the derivatives with a factor composed of ρV_h^2 and the appropriate power of a representative length, where V_h is the steady lineal speed of the discrete lump, and is given by

$$u_d^{DOo} + P_{dd}^{do} x_d^{L_{\mathscr{D}}D} = V_h \begin{bmatrix} \sqrt{\cos^2\alpha_h + \cos^2\beta_h} \\ \sin\beta_h \\ \sin\alpha_h \end{bmatrix} \tag{97}$$

where α_h and β_h are respectively the angles of downslip* and of sideslip for the discrete lump.

The intermediate generalised aerodynamic force on the discrete lump in the datum state may be obtained either by theoretical calculation or experimental measurement.

Both the aerodynamic derivatives and the intermediate generalised aerodynamic force in the datum state will be functions of the local Mach number (and, in principle, of Reynolds number) and of the angles of downslip and sideslip α_h and β_h .

Insofar as aerodynamic theories do not usually cater for fore-and-aft motion, motion, it can be assumed that derivatives $\dfrac{\partial z_{ah}}{\partial q_{kl}^*}$ and $\dfrac{\partial z_{ah}}{\partial \dot{q}_{kl}^*}$ are zero, with the exception of $\dfrac{\partial z_{ah}}{\partial \dot{q}_{hl}^*}$, which is usually assumed to be given approximately by

* See Ref 3, section 6.2.

$$\frac{\partial z_{ah}}{\partial \dot{q}^*_{hl}} \simeq \frac{2}{V_h} z_{ah_{\mathscr{D}}} \quad . \tag{98}$$

Multiplying equation (96) by $\tilde{H}_h$, and using equations (79) and (27), we obtain the generalised aerodynamic force z_a on the aircraft as a function of the generalised co-ordinates q in the form given by equation (93), where, for the discrete-lump approach that we have just considered,

$$z_{a_{\mathscr{E}}} = \sum_{LA} \tilde{H}_h z_{ah_{\mathscr{D}}} - C_a q_{\mathscr{E}} \tag{99}$$

$$C_a = - \sum_{LA} \tilde{H}_h \frac{\partial z_{ah}}{\partial q^*_k} H_k \tag{100}$$

$$B_a = - \sum_{LA} \tilde{H}_h \frac{\partial z_{ah}}{\partial \dot{q}^*_k} H_k \quad . \tag{101}$$

It is necessary to relate the elements of the aerodynamic stiffness and damping matrices C_a and B_a which correspond to whole-body degrees of freedom to the aerodynamic derivatives which the flight dynamicist traditionally uses, for a wealth of knowledge exists on such derivatives. Now the flight dynamicist usually writes the resolutes in the undeformed-body axes directions b of the aerodynamic force on an aircraft as a whole, as

$$\left.\begin{aligned} X &= X_e + \left(X_u u' + X_v v' + \ldots + X_r r'\right) + \left(X_{\dot{u}} \dot{u}' + X_{\dot{v}} \dot{v}' + \ldots + X_{\dot{r}} \dot{r}'\right) \\ Y &= Y_e + \left(Y_u u' + Y_v v' + \ldots + Y_r r'\right) + \left(Y_{\dot{u}} \dot{u}' + Y_{\dot{v}} \dot{v}' + \ldots + Y_{\dot{r}} \dot{r}'\right) \\ Z &= Z_e + \left(Z_u u' + Z_v v' + \ldots + Z_r r'\right) + \left(Z_{\dot{u}} \dot{u}' + Z_{\dot{v}} \dot{v}' + \ldots + Z_{\dot{r}} \dot{r}'\right) \end{aligned}\right\} \tag{102}$$

and the resolutes in the undeformed-body axes directions b of the aerodynamic moment about the origin B on an aircraft as a whole, as

$$\left.\begin{aligned} L &= L_e + \left(L_u u' + L_v v' + \ldots + L_r r'\right) + \left(L_{\dot{u}} \dot{u}' + L_{\dot{v}} \dot{v}' + \ldots + L_{\dot{r}} \dot{r}'\right) \\ M &= M_e + \left(M_u u' + M_v v' + \ldots + M_r r'\right) + \left(M_{\dot{u}} \dot{u}' + M_{\dot{v}} \dot{v}' + \ldots + M_{\dot{r}} \dot{r}'\right) \\ N &= N_e + \left(N_u u' + N_v v' + \ldots + N_r r'\right) + \left(N_{\dot{u}} \dot{u}' + N_{\dot{v}} \dot{v}' + \ldots + N_{\dot{r}} \dot{r}'\right) \end{aligned}\right\} \quad (103)$$

(see Ref 3, section 10.2.1).

As mentioned in section 6.2, the flight dynamicist's perturbation quantities are related to ours by the following identities:

$$u' = u - u_e \qquad v' = v - v_e \qquad w' = w - w_e \qquad (104)$$

where

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} \equiv u_b^{BOo} \qquad \begin{bmatrix} u_e \\ v_e \\ w_e \end{bmatrix} \equiv u_d^{DOo} \qquad (105)$$

and

$$p' = p - p_e \qquad q' = q - q_e \qquad r' = r - r_e \qquad (106)$$

where

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} \equiv p_b^{bo} \qquad \begin{bmatrix} p_e \\ q_e \\ r_e \end{bmatrix} \equiv p_d^{do} \quad . \qquad (107)$$

For simple harmonic oscillations of frequency ν , and, in the usual way, leaving out the time-dependence factor $e^{i\nu t}$, equations (58) and (66) give for the relationships between the flight dynamicist's perturbation quantities u', v', w', p', q', r' and our perturbation quantities q_1, q_2, q_3, q_4, q_5, q_6

$$\begin{bmatrix} u' \\ v' \\ w' \end{bmatrix} \doteq \left(i\nu + P_{dd}^{do}\right) \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + U_{dd}^{DOo} \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} \qquad (108)$$

$$\begin{bmatrix} p' \\ q' \\ r' \end{bmatrix} \simeq \left(i\nu + P_{dd}^{do}\right) \begin{bmatrix} q_4 \\ q_5 \\ q_6 \end{bmatrix} \tag{109}$$

which may be combined to give

$$\begin{bmatrix} u' \\ v' \\ w' \\ p' \\ q' \\ r' \end{bmatrix} \simeq \left(i\nu + \left[\begin{array}{c|c} P_{dd}^{do} & U_{dd}^{DOo} \\ \hline 0 & P_{dd}^{do} \end{array}\right] \right) \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} \tag{110}$$

where 0 is the 3 × 3 null matrix.

The flight dynamicist's forces are related to ours by the following identities:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv S_{bd} \begin{bmatrix} z_{a1} \\ z_{a2} \\ z_{a3} \end{bmatrix} \qquad \begin{bmatrix} L \\ M \\ N \end{bmatrix} \equiv S_{bd} \begin{bmatrix} z_{a4} \\ z_{a5} \\ z_{a6} \end{bmatrix} \tag{111}$$

$$\begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix} \equiv \begin{bmatrix} z_{a1_{\&}} \\ z_{a2_{\&}} \\ z_{a3_{\&}} \end{bmatrix} \qquad \begin{bmatrix} L_e \\ M_e \\ N_e \end{bmatrix} \equiv \begin{bmatrix} z_{a4_{\&}} \\ z_{a5_{\&}} \\ z_{a6_{\&}} \end{bmatrix} \quad . \tag{112}$$

By using equations (6), (16) and (112), equation (111) may be written, to first order, as

$$\begin{bmatrix} z_{a1} \\ z_{a2} \\ z_{a3} \end{bmatrix} \simeq \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} 0 & -Z_e & Y_e \\ Z_e & 0 & -X_e \\ -Y_e & X_e & 0 \end{bmatrix} \phi^{bd} \tag{113}$$

$$\begin{bmatrix} z_{a4} \\ z_{a5} \\ z_{a6} \end{bmatrix} \doteq \begin{bmatrix} L \\ M \\ N \end{bmatrix} - \begin{bmatrix} 0 & -N_e & M_e \\ N_e & 0 & -L_e \\ -M_e & L_e & 0 \end{bmatrix} \phi^{bd} \quad . \tag{114}$$

Therefore, substituting from equations (102) and (103) into equations (113) and (114), we obtain, for the generalised aerodynamic forces in the whole-body degrees of freedom in terms of the flight dynamicist's aerodynamic derivatives,

$$\begin{bmatrix} z_{a1} \\ z_{a2} \\ z_{a3} \end{bmatrix} \doteq \begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix} + \begin{bmatrix} X_u & X_v & \dots X_r \\ Y_u & Y_v & \dots Y_r \\ Z_u & Z_v & \dots Z_r \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \\ p' \\ q' \\ r' \end{bmatrix} + \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & \dots X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & \dots Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & \dots Z_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dot{u}' \\ \dot{v}' \\ \dot{w}' \\ \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} - \begin{bmatrix} 0 & -Z_e & Y_e \\ Z_e & 0 & -X_e \\ -Y_e & X_e & 0 \end{bmatrix} \phi^{bd}$$

...... (115)

$$\begin{bmatrix} z_{a4} \\ z_{a5} \\ z_{a6} \end{bmatrix} \doteq \begin{bmatrix} L_e \\ M_e \\ N_e \end{bmatrix} + \begin{bmatrix} L_u & L_v & \dots L_r \\ M_u & M_v & \dots M_r \\ N_u & N_v & \dots N_r \end{bmatrix} \begin{bmatrix} u' \\ v' \\ w' \\ p' \\ q' \\ r' \end{bmatrix} + \begin{bmatrix} L_{\dot{u}} & L_{\dot{v}} & \dots L_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & \dots M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & \dots N_{\dot{r}} \end{bmatrix} \begin{bmatrix} \dot{u}' \\ \dot{v}' \\ \dot{w}' \\ \dot{p}' \\ \dot{q}' \\ \dot{r}' \end{bmatrix} - \begin{bmatrix} 0 & -N_e & M_e \\ N_e & 0 & -L_e \\ -M_e & L_e & 0 \end{bmatrix} \phi^{bd} \quad .$$

...... (116)

Hence, utilizing the relationships (112) and (110), we see from equations (115) and (116) that the expressions for the elements of the matrices C_a and B_a in equation (93) associated with whole-body degrees of freedom are given by

$$C_{a_{WW}} = \nu^2 \begin{bmatrix} X_{\dot{u}} X_{\dot{v}} & \dots & X_{\dot{r}} \\ Y_{\dot{u}} Y_{\dot{v}} & \dots & Y_{\dot{r}} \\ \dots & \dots & \dots \\ N_{\dot{u}} N_{\dot{v}} & \dots & N_{\dot{r}} \end{bmatrix} - \begin{bmatrix} X_u X_v & \dots & X_r \\ Y_u Y_v & \dots & Y_r \\ \dots & \dots & \dots \\ N_u N_v & \dots & N_r \end{bmatrix} \left[\begin{array}{c|c} P^{do}_{dd} & U^{DOo}_{dd} \\ \hline 0 & P^{do}_{dd} \end{array}\right] + \left[\begin{array}{c|ccc} & 0 & -Z_e & Y_e \\ 0 & Z_e & 0 & -X_e \\ & -Y_e & X_e & 0 \\ \hline & 0 & -N_e & M_e \\ 0 & N_e & 0 & -L_e \\ & -M_e & L_e & 0 \end{array}\right] \tag{117}$$

$$B_{a_{WW}} = -\begin{bmatrix} X_u X_v & \ldots & X_r \\ Y_u Y_v & \ldots & Y_r \\ \ldots & \ldots & \ldots \\ N_u N_v & \ldots & N_r \end{bmatrix} - \begin{bmatrix} X_{\dot u} X_{\dot v} & \ldots & X_{\dot r} \\ Y_{\dot u} Y_{\dot v} & \ldots & Y_{\dot r} \\ \ldots & \ldots & \ldots \\ N_{\dot u} N_{\dot v} & \ldots & N_{\dot r} \end{bmatrix} \left[\begin{array}{c|c} P^{do}_{dd} & U^{DOo}_{dd} \\ \hline 0 & P^{do}_{dd} \end{array}\right] . \tag{118}$$

Equations (117) and (118), with equations (94) and (95), enable the aerodynamic coefficients in the aerodynamic stiffness and damping sub-matrices for the whole-body degrees of freedom, as used by the structural dynamicist, to be determined in terms of so-called aerodynamic derivatives as used by the flight dynamicist. Additionally we note that aerodynamic stiffness matrix C_a depends upon aerodynamic forces on the aircraft in the datum motion.

8 EQUATIONS OF MOTION

8.1 Equation for a particle

We postulate that the earth frame of reference is an inertial frame of reference. Strictly this is not true, but since the angular velocities of the earth about its own axis and about the sun are very small compared with the quotient of the lineal velocity of an aircraft divided by a typical length, it is a reasonable approximation. Hence, on the postulate that the earth frame of reference is an inertial frame approximately, it is a frame in which Newton's laws apply. Accordingly the equation of motion for a particle is

$$mf^{POo}_d - k_d = 0 \tag{119}$$

(see Ref 6, article 8.14, equation (1)) where f^{POo}_d is given by equation (51).

8.2 Equation for the aircraft

Premultiplying equation (119) by $\tilde{H}_h \tilde{J}_h$, summing over all particles of the aircraft, and using equations (51), (32), (33) and (74), we obtain the equation of motion for the aircraft in terms of generalised co-ordinates and forces

$$y_D + y_{f_g} + A_e\ddot{q} + B_c\dot{q} + C_f q - z_s - z_g - z_a \simeq 0 \tag{120}$$

where y_D is an N × 1 column matrix of generalised effective forces associated with acceleration of the datum-motion origin, given by

$$y_D \triangleq \left(\sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m\right\}\right) P_{dd}^{do} u_d^{DOo} \qquad (121)$$

$y_{f_{\mathcal{E}}}$ is an $N \times 1$ column matrix of generalised centrifugal forces associated with the acceleration of a particle relative to the datum-motion frame of reference in the datum-motion, given by

$$y_{f_{\mathcal{E}}} \triangleq \sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m\right\} P_{dd}^{do^2} x_d^{L\mathscr{D}D} + \sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m P_{dd}^{do^2} x_\ell^{PL}\right\}$$

$$+ \left(\sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m P_{dd}^{do^2} J_h\right\} H_h\right) q_{\mathcal{E}} \qquad (122)$$

$A_e\ddot{q}$ are the generalised Newtonian forces due to the perturbation, where A_e is an $N \times N$ matrix given by

$$A_e = \sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m J_h\right\} H_h \qquad (123)$$

$B_c\dot{q}$ are the generalised Coriolis forces due to the perturbation, where B_c is an $N \times N$ matrix given by

$$B_c = \sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h 2mP_{dd}^{do} J_h\right\} H_h \qquad (124)$$

$C_f q$ are the generalised centrifugal forces due to the perturbation, where C_f is an $N \times N$ matrix given by

$$C_f = \sum_{LA} \tilde{H}_h \left\{\sum_{PL} \tilde{J}_h m P_{dd}^{do^2} J_h\right\} H_h \qquad (125)$$

and z_s, z_g, z_a are $N \times 1$ column matrices of respectively the generalised structural, gravitational and aerodynamic applied forces, given respectively by equations (83), (92) and (93). In some problems it may be convenient to split the aerodynamic force into an external aerodynamic force associated with the external airflow over the aircraft, and an internal aerodynamic, or 'propulsive', 010

force associated with the internal flow through the aircraft. When the aircraft is in contact with the ground there will of course be additional ground reaction forces, which can readily be included.

Lagrange showed that the term $A_e\ddot{q}$ in equation (120), may be written as $\frac{d}{dt}\left(\frac{\partial\tilde{T}}{\partial\dot{q}}\right) - \frac{\partial\tilde{T}}{\partial q}$ where T is the kinetic energy of the aircraft relative to the datum-motion frame of reference (see Ref 6, article 8.14). However, in order to evaluate this Lagrangian expression one has first to evaluate the velocity $\dot{x}_d^{PP\mathscr{D}}$ of a particle, which, by equations (23) and (31), is given by

$$\dot{x}_d^{PP\mathscr{D}} \triangleq J_h\dot{q}_h^* \quad . \tag{126}$$

To do this, one needs to form the matrix J_h , which means that one is in a position to evaluate the inner summation $\left\{\sum_{PL} \tilde{J}_h m J_h\right\}$ in equation (123) directly, without determining T .

Equation (120) represents the formal equation of motion. Now the mass m of a particle of the aircraft is a function of its position, as represented by x_ℓ^{PL} , within the discrete lump to which it belongs. By equation (24) J_h is also a function of x_ℓ^{PL} . Therefore in equations (121), (122), (123), (124) and (125) the expressions in curly brackets are functions of the position x_ℓ^{PL} of each particle within the discrete lump to which it belongs, and of the angular velocity P_{dd}^{do} of the aircraft in the datum motion. The first stage in a calculation is therefore to do the summations $\sum_{PL}$ over all the particles of each discrete lump. The summations are usually done by integration. For example, temporarily, for brevity, writing $x_{\ell_1}^{PL}$ as ξ , $x_{\ell_2}^{PL}$ as η , $x_{\ell_3}^{PL}$ as ζ , the expression in curly brackets on the right-hand side of equation (123) for A_e , which is a 6×6 matrix, and usually denoted by just A by structural dynamicists, may be written

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$$
\begin{bmatrix}
\int m & 0 & 0 & 0 & \int m\zeta & -\int m\eta \\
0 & \int m & 0 & -\int m\zeta & 0 & \int m\xi \\
0 & 0 & \int m & \int m\eta & -\int m\xi & 0 \\
0 & -\int m\zeta & \int m\eta & \int m(\eta^2+\zeta^2) & -\int m\xi\eta & -\int m\zeta\xi \\
\int m\zeta & 0 & -\int m\xi & -\int m\xi\eta & \int m(\zeta^2+\xi^2) & -\int m\eta\zeta \\
-\int m\eta & \int m\xi & 0 & -\int m\zeta\xi & -\int m\eta\zeta & \int m(\xi^2+\eta^2)
\end{bmatrix}
\qquad (127)
$$

where $\int$ indicates integration over all particles of the discrete lump.

The expressions in curly brackets are therefore functions of the discrete lumps, and depend upon their position, as represented by $x_d^{L\mathscr{D}}$, within the aircraft as a whole. From section 5.2, H_h is also a function of $x_d^{L\mathscr{D}}$. Hence, once the expression in curly brackets have been evaluated, one is able to do the summations $\sum_{LA}$ over all the discrete lumps of the aircraft, and thereby determine the coefficients of all the terms in equation (120).

Often, the generalised co-ordinates will be chosen to be normal co-ordinates. This entails appropriate determination of the modal shape matrices H_h, and it has the effect of making the inertia matrix A_e diagonal.

8.3 Equation for the datum motion

From equation (119) we have, as the equation for the datum motion,

$$y_D + y_{f_g} - z_{s_g} - z_{g_g} - z_{a_g} \bumpeq 0 \qquad (128)$$

where y_D, y_{f_g}, z_{s_g}, z_{g_g}, z_{a_g} are given by equations (121), (122), (84), (92), and (99) respectively.

Now the datum motions that we have restricted ourselves to, are, as stated in section 3, those in which the lineal velocity resolutes u_d^{DOo} and angular velocity resolutes p_d^{do} are constant. Moreover, as pointed out in section 7.3, in order that the gravitational forces shall remain constant in the datum motion, we must restrict ourselves to datum motions in which the angular velocity of the aircraft relative to the earth frame of reference is about a vertical axis, that is, that

$$p_o^{do} = \begin{bmatrix} 0 \\ 0 \\ p_{o_3}^{do} \end{bmatrix} \tag{129}$$

where $p_{o_3}^{do}$ is the constant angular velocity about the vertical axis o_3 .

The datum motions that we are restricting ourselves to are, therefore, expressing them as generally as we can, constant lineal speed, constant angular velocity, climbing turns. Naturally this restriction embraces constant lineal speed straight climbing flight as a special case.

In these datum motions the angles of bank ϕ_1^{do} and of inclination ϕ_2^{do} will clearly be constant, but the heading angle ϕ_3^{do} will increase linearly with time, as given by

$$\phi_3^{do} = p_{o_3}^{do} t \tag{130}$$

where we are assuming that the heading is zero at time zero.

From equation (5), the orientation matrix S_{do} is therefore given by

$$S_{do} = \begin{bmatrix} \cos\phi_2^{do}\cos p_{o_3}^{do}t & \cos\phi_2^{do}\sin p_{o_3}^{do}t & -\sin\phi_2^{do} \\ \begin{matrix}\sin\phi_1^{do}\sin\phi_2^{do}\cos p_{o_3}^{do}t \\ -\cos\phi_1^{do}\sin p_{o_3}^{do}t\end{matrix} & \begin{matrix}\sin\phi_1^{do}\sin\phi_2^{do}\sin p_{o_3}^{do}t \\ +\cos\phi_1^{do}\cos p_{o_3}^{do}t\end{matrix} & \sin\phi_1^{do}\cos\phi_2^{do} \\ \begin{matrix}\cos\phi_1^{do}\sin\phi_2^{do}\cos p_{o_3}^{do}t \\ +\sin\phi_1^{do}\sin p_{o_3}^{do}t\end{matrix} & \begin{matrix}\cos\phi_1^{do}\sin\phi_2^{do}\sin p_{o_3}^{do}t \\ -\sin\phi_1^{do}\cos p_{o_3}^{do}t\end{matrix} & \cos\phi_1^{do}\cos\phi_2^{do} \end{bmatrix}$$

...... (131)

and therefore that the angular velocity of the aircraft resolved with respect to the datum-motion axes directions d is given by

$$p_d^{do} = p_{o_3}^{do} \begin{bmatrix} -\sin\phi_2^{do} \\ \sin\phi_1^{do}\cos\phi_2^{do} \\ \cos\phi_1^{do}\cos\phi_2^{do} \end{bmatrix} . \tag{132}$$

The lineal velocity of the aircraft in the datum motion resolved with respect to the earth axes directions o may be written (see Fig 2) as

$$u_o^{DOo} = V \begin{bmatrix} \cos\gamma\cos\left(p_{o_3}^{do}t + \delta\right) \\ \cos\gamma\sin\left(p_{o_3}^{do}t + \delta\right) \\ -\sin\gamma \end{bmatrix} \tag{133}$$

where V is the steady lineal speed of the aircraft

γ is the angle of climb of the aircraft

δ is the angle of drift of the aircraft (that is, the angle of track minus the heading angle).

By using equations (131) and (A-3), equation (133) shows that the lineal velocity of the aircraft resolved with respect to the datum-motion axes directions d is given by

$$u_d^{DOo} = V \begin{bmatrix} \cos\gamma\cos\phi_2^{do}\cos\delta + \sin\gamma\sin\phi_2^{do} \\ \cos\gamma\sin\phi_1^{do}\sin\phi_2^{do}\cos\delta + \cos\gamma\cos\phi_1^{do}\sin\delta - \sin\gamma\sin\phi_1^{do}\cos\phi_2^{do} \\ \cos\gamma\cos\phi_1^{do}\sin\phi_2^{do}\cos\delta + \cos\gamma\sin\phi_1^{do}\sin\delta - \sin\gamma\cos\phi_1^{do}\cos\phi_2^{do} \end{bmatrix} .$$

...... (134)

Now, in a given problem, the datum-motion conditions that one will usually be given are

the lineal speed of the aircraft, V

the angle of climb of the aircraft, γ

the angular velocity of the aircraft, $p_{o_3}^{do}$

with, additionally, the requirement that the aircraft is not sideslipping, that is that

$$\cos\gamma \sin\phi_1^{do} \sin\phi_2^{do} \cos\delta + \cos\gamma \cos\phi_1^{do} \sin\delta - \sin\gamma \sin\phi_1^{do} \cos\phi_2^{do} = 0 \; . \qquad \ldots\ldots\ (135)$$

The angle of bank ϕ_1^{do}, the angle of inclination ϕ_2^{do}, and the angle of drift δ need to be determined, as do the settings of the throttle, aileron, elevator and rudder that are necessary to trim the aircraft in speed, roll, pitch and yaw. These seven quantities, together with the (N - 6) quantities which define the static deformations, as given by the column matrix $q_{\&}$, can in principle be determined from the N equilibrium equations (128) together with the sideslip equation (135). The settings of throttle, aileron, elevator and rudder will affect the contributions of one or two of the discrete lumps only in each case, and mainly through the contributions to the generalised aerodynamic forces $z_{a_{\&}}$.

The best method of solving the equations (128) and (135) will depend upon the problem. In some problems, for example it may be found simpler in practice to take the angles of bank ϕ_1^{do} and of inclination ϕ_2^{do} as given, and to solve the equations for the lineal speed V and the angular velocity $p_{o_3}^{do}$ instead, *and then to proceed to the desired values of* V *and* $p_{o_3}^{do}$ by a process of iteration. But these are matters which it is not proposed to go into further here.

8.4 Equation for the perturbed motion

From equation (119) we have, as the equation for the perturbed motion,

$$A_e\ddot{q} + (B_c + B_s + B_a)\dot{q} + (C_f + C_s + C_a)q \triangleq o \qquad (136)$$

where A_e, B_c, B_s, B_a, C_f, C_s, C_a, are given by equations (123), (124), (83), (101), (125), (80) and (100) respectively.

Equation (136) may be written in partitioned form as

$$\left[\begin{array}{c|c} A_{WW} & A_{WF} \\ \hline A_{FW} & A_{FF} \end{array}\right]\left[\begin{array}{c} \ddot{q}_W \\ \hline \ddot{q}_F \end{array}\right] + \left[\begin{array}{c|c} B_{WW} & B_{WF} \\ \hline B_{FW} & B_{FF} \end{array}\right]\left[\begin{array}{c} \dot{q}_W \\ \hline \dot{q}_F \end{array}\right] + \left[\begin{array}{c|c} C_{WW} & C_{WF} \\ \hline C_{FW} & C_{FF} \end{array}\right]\left[\begin{array}{c} q_W \\ \hline q_F \end{array}\right] = o \qquad (137)$$

where subscripts W, F denote submatrices associated with the whole-body and deformational degrees of freedom respectively.

Now an assumption that is often made is that, for motion at frequencies which are small compared with the frequencies of the deformational modes, the terms involving $\ddot{q}_F$ and $\dot{q}_F$ may be neglected.

With this assumption, the lower submatrix equation of the matrix equation (137) gives

$$C_{FF}q_F = -\left(A_{FW}\ddot{q}_W + B_{FW}\dot{q}_W + C_{FW}q_W\right) \quad . \tag{138}$$

Hence, substituting for q_F from equation (138) into the upper submatrix equation of the matrix equation (137), we obtain

$$\left(A_{WW} - C_{WF}C_{FF}^{-1}A_{FW}\right)\ddot{q}_W + \left(B_{WW} - C_{WF}C_{FF}^{-1}B_{FW}\right)\dot{q}_W + \left(C_{WW} - C_{WF}C_{FF}^{-1}C_{FW}\right)q_W = o \quad . \tag{139}$$

We note that, had we ignored the deformational degrees of freedom entirely, equation (136) would have reduced to the following simplified equation involving the whole-body degrees of freedom only

$$A_{WW}\ddot{q}_W + B_{WW}\dot{q}_W + C_{WW}q_W = o \quad . \tag{140}$$

Equation (139) may therefore be interpreted as an improvement on equation (140) in that the coefficients are 'modified' in a way which takes account of the static aeroelastic properties of the system, for the basic assumption made earlier amounts to saying that at all times the structure is in static equilibrium under the forces appropriate to the deformed state at each instant of the perturbed motion.

Appendix A

LAWS REPRESENTING WELL-KNOWN RELATIONSHIPS

If v and y are *any* vectors, and if their representations as column matrices are v_d and y_d and as associated antisymmetric matrices are V_{dd} and Y_{dd}, then the antisymmetric matrix that is associated with the column matrix $Y_{dd}v_d$ is $Y_{dd}V_{dd} - V_{dd}Y_{dd}$. It follows therefore that

$$Y_{dd}v_d + V_{dd}y_d = o \tag{A-1}$$

$$V_{dd}v_d = o \tag{A-2}$$

where o is the null column matrix. (See Ref 6, article 8.5, equation (5).)

The laws for change of directions of resolution, or axes, for *any* vector v are

$$\left.\begin{aligned} v_d &= S_{do}v_o \\ V_{dd} &= S_{do}V_{oo}S_{od} \quad . \end{aligned}\right\} \tag{A-3}$$

From the vector law of addition we obtain the position of a point P relative to an origin D in terms of the position relative to an origin O

$$\left.\begin{aligned} x_o^{PD} &= x_o^{PO} - x_o^{DO} \\ X_{oo}^{PD} &= X_{oo}^{PO} - X_{oo}^{DO} \quad . \end{aligned}\right\} \tag{A-4}$$

Combining equation (A-4) with the *laws for change of axes* (A-3), we obtain the position of a point P relative to an origin D and resolved with respect to axes d in terms of the position relative to an origin O and resolved with respect to axes o

$$\left.\begin{aligned} x_d^{PD} &= S_{do}\left(x_o^{PO} - x_o^{DO}\right) \\ X_{dd}^{PD} &= S_{do}\left(X_{oo}^{PO} - X_{oo}^{DO}\right)S_{od} \quad . \end{aligned}\right\} \tag{A-5}$$

The orientation of a set of orthogonal directions ℓ relative to a set o may be expressed in terms of the orientation relative to a set d by the law of inner multiplication

$$S_{\ell o} = S_{\ell d} S_{do} \quad . \tag{A-6}$$

By differentiating equation (A-5) we can obtain the lineal velocity of a point P relative to a frame of reference Dd in terms of the lineal velocity relative to a frame of reference Oo, all resolved with respect to some axes b

$$\left.\begin{aligned} u_b^{PDd} &= u_b^{POo} - u_b^{DOo} - P_{bb}^{do}\left(x_b^{PO} - x_b^{DO}\right) \\ U_{bb}^{PDd} &= U_{bb}^{POo} - U_{bb}^{DOo} - P_{bb}^{do}\left(X_{bb}^{PO} - X_{bb}^{DO}\right) + \left(X_{bb}^{PO} - X_{bb}^{DO}\right)P_{bb}^{do} \end{aligned}\right\} \tag{A-7}$$

(see Ref 6, article 8.7, equation (1)).

By differentiating equation (A-6), we obtain the vector law of addition for angular velocity, whereby the angular velocity of a set of orthogonal directions ℓ relative to a set d is given in terms of the angular velocity relative to a set o, all resolved with respect to some axes b, by

$$\left.\begin{aligned} P_b^{\ell d} &= P_b^{\ell o} - P_b^{do} \\ P_{bb}^{\ell d} &= P_{bb}^{\ell o} - P_{bb}^{do} \quad . \end{aligned}\right\} \tag{A-8}$$

Appendix B

COMPARISON OF SYMBOLS WITH THOSE USED BY OTHER AUTHORS

	This Report	Hopkin (Ref 3)	Woodcock (Ref 5)	Frazer, Duncan & Collar (Ref 6)	Baldock
Origin of earth frame of reference	O	O_o	O_0	O	
Origin of datum-motion frame of reference	D		O_c	O	R_o
Origin of undeformed-body frame of reference	B	O	O_n		A
Origin of discrete-lump frame of reference	L				
Axes directions of earth frame of reference	o	o	O		
Axes directions of datum-motion frame of reference	d		c		
Axes directions of undeformed-body frame of reference	b		n		
Axes direction of discrete-lump frame of reference	ℓ				
Point occupied by a particle	P			P	
Position of a particle relative to the datum-motion origin and resolved along the datum-motion axes	x_d^{PD}		$x_c^{(c)}$	x	
Position of a particle relative to the datum-motion origin and resolved along the undeformed-body axes	x_b^{PD}		$x_c^{(n)}$		

	This Report	Hopkin (Ref 3)	Woodcock (Ref 5)	Frazer, Duncan & Collar (Ref 6)	Baldock
Unperturbed position of a particle relative to the datum-motion origin and resolved along the datum-motion axes	$x_d^{P\mathscr{D}D}\left(X_{dd}^{P\mathscr{D}D}\right)$		$x_f\left({}^{A}x_f\right)$		
Position of the undeformed-body origin relative to the datum-motion origin and resolved along the datum-motion axes	x_d^{BD}		$x_1^{(c)}$		
Position of a particle relative to the undeformed-body origin and resolved along the datum-motion axes	x_d^{PB}		$x_n^{(c)}$		
Position of a particle relative to the undeformed-body origin and resolved along the undeformed-body axes	x_b^{PB}		$x_n^{(n)}$		
Position of a particle relative to the discrete-lump origin and resolved along the discrete-lump axes	x_ℓ^{PL}				
Position of the discrete-lump origin relative to its datum position and resolved along the datum-motion axes	$x_d^{LL\mathscr{D}}$				$\{a_3, .., a_2\}$
Position of the datum position of the discrete-lump origin relative to the datum-motion origin and resolved along the datum-motion axes	$x_d^{L\mathscr{D}D}$				$\{x_3, .., x_2\}$
Orientation of the datum-motion axes relative to the earth axes	S_{do}	S_{Φ_e}	S_{Φ_f}		
Orientation of the undeformed-body axes relative to the datum-motion axes	S_{bd}	S_{ϕ}	S		
Orientation of the discrete-lump axes relative to the datum-motion axes	$S_{\ell d}$				

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	This Report	Hopkin (Ref 3)	Woodcock (Ref 5)	Frazer, Duncan & Collar (Ref 6)	Baldock
Angular rotation of the datum-motion axes from the earth axes	ϕ^{do}	Φ_e	Φ_f		
Angular rotation of the undeformed-body axes from the datum-motion axes	$\phi^{bd}(\Phi^{bd})$	ϕ	$\phi(A_\phi)$		
Angular rotation of the discrete-lump axes from the datum-motion axes	$\phi^{\ell d}$				$\{.., \alpha, ..\}$
Lineal velocity of a particle relative to the earth frame of reference and resolved along the earth axes	u_o^{POo}		$u_{m0}^{(O)}$		
Lineal velocity of a particle relative to the earth frame of reference and resolved along the datum-motion axes	u_d^{POo}		$u_{m0}^{(c)}$	υ	
Lineal velocity of the datum-motion origin relative to the earth frame of reference and resolved along the datum-motion axes	u_d^{DOo}	u_e	u_f	u	
Lineal velocity of the undeformed-body origin relative to the earth frame of reference and resolved along the undeformed-body axes	u_b^{BOo}	u	u		
Lineal velocity of the discrete-lump origin relative to a frame of reference having an origin the datum-state position of the discrete-lump origin and the datum-motion axes directions and resolved along the datum-motion axes	$u_d^{LL_{\mathscr{D}}d}$				$\{.., .., V\gamma\}$
Angular velocity of the datum-motion axes relative to the earth axes and resolved along the datum-motion axes	$p_d^{do}\left(p_{dd}^{do}\right)$	p_e		$p(\varpi)$	

	This Report	Hopkin (Ref 3)	Woodcock (Ref 5)	Frazer, Duncan & Collar (Ref 6)	Baldock
Angular velocity of the undeformed-body axes relative to the earth axes and resolved along the undeformed-body axes	p_b^{bo}	p	P		
Lineal acceleration of a particle relative to the earth frame of reference and resolved along the datum-motion axes	f_d^{POo}			α	
Applied force on a particle resolved along the datum-motion axes	k_d		$\bar{e}^{(c)}$	X	
Applied force on a particle in the datum-motion resolved along the datum-motion axes	$k_{d_{\&}}$		$\bar{e}_f$		
Generalised co-ordinate	q		q_i	q_i	q
Total generalised applied force	z		$\bar{Q}_i$	P_i	Q_r
Generalised structural force	z_s		$- E_i$		
Generalised aerodynamic force	z_a		Q_i		
Modal shape column matrix function	$\mathscr{F}$			f	
Modal shape matrix coefficient	$J_h H_h$		R		{e f F}
Mass of a particle	m		δm	m	
Kinetic energy of the aircraft relative to the datum-motion frame of reference	T		W	T	T
Elastic strain energy	W				U

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	This Report	Hopkin (Ref 3)	Woodcock (Ref 5)	Frazer, Duncan & Collar (Ref 6)	Baldock
Matrix of inertia coefficients for the aircraft	A_e		A_{ij}	A	A
Matrix of aerodynamic damping coefficients for the aircraft	B_a				ρVB
Matrix of gravitational stiffness coefficients for the aircraft	C_g		G_{ij}		
Matrix of structural stiffness coefficients for the aircraft	C_s		E_{ij}	E	E
Matrix of aerodynamic stiffness coefficients for the aircraft	C_a		$-Q_{ij}$ (closely)	W	$\rho V^2 C$

LIST OF SYMBOLS

Basic symbols

[Note: Lower case letters represent vector, column matrix or scalar quantities. Upper case letters usually represent square or rectangular matrices]

A	$N \times N$ matrix of inertia coefficients for the aircraft	
B	$N \times N$ matrix of damping coefficients for the aircraft	
C	$N \times N$ matrix of stiffness coefficients for the aircraft	
f	(3×1) lineal acceleration	
$\mathscr{F}$	3×1 modal shape column matrix function defined by equation (9)	
g	(3×1) acceleration due to gravity, and its scalar magnitude	
H_h	$6 \times N$ modal shape matrix coefficient defined by equation (26)	
$\mathcal{H}_h$	6×1 modal shape column matrix function defined by equation (25)	
I	3×3 unit matrix	
J_h	3×6 matrix coefficient for the h^{th} discrete lump defined by equation (24)	
k	(3×1) applied force on a particle	
m	mass of a particle	
n	number of discrete lumps	
N	number of generalised co-ordinates	
o	column null matrix	
O	3×3 null matrix	
p	(3×1)	angular velocity
P	$(3 \times 3$ antisymmetric$)$	
q	$(N \times 1)$ generalised co-ordinate	
q_h^*	(6×1) intermediate generalised co-ordinate for the h^{th} discrete lump	
S	3×3 orientation matrix	
t	time	
T	kinetic energy of the aircraft relative to the datum-motion frame of reference	
u	(3×1)	lineal velocity
U	$(3 \times 3$ antisymmetric$)$	
v	(3×1)	any vector quantity
V	$(3 \times 3$ antisymmetric$)$	
V	lineal speed of the aircraft	
V_h	lineal speed of a discrete lump	
W	elastic strain energy	

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LIST OF SYMBOLS (continued)

x	(3 × 1)	lineal position
X	(3 × 3 antisymmetric)	lineal position
y	(N × 1) generalised effective force	
y	(3 × 1)	any vector quantity
Y	(3 × 3 antisymmetric)	any vector quantity
z	(N × 1) generalised applied force	
z_h	(6 × 1) intermediate generalised applied force	
α_h	angle of downslip of a discrete lump (see Ref 3, section 6.2)	
β_h	angle of sideslip of a discrete lump	
γ	angle of climb	
δ	angle of drift	
Δ	total work done in an infinitesimal displacement	
ν	frequency of oscillatory perturbation	
ρ	ambient air density	
ϕ	(3 × 1)	angular rotation
Φ	(3 × 3 antisymmetric)	angular rotation
ω	(N × 1) modal frequency	
$\hat{\Omega}$	N × N diagonal matrix of modal frequencies	

Subscripts

a	aerodynamic
A	aircraft
b	undeformed-body axes directions
$\mathcal{B}$	undeformed-body state value or position
c	Coriolis
d	datum-motion axes directions
D	datum-motion origin
$\mathcal{D}$	datum state value or position
e	Newtonian
$\mathcal{E}$	datum-motion or equilibrium state value or position
f	centrifugal
F	deformational
g	gravitational
h	a specific discrete lump (h = 1, 2, ..., n)
i	a specific intermediate generalised co-ordinate of the h^{th} discrete lump (i = 1, 2, ..., 6)

LIST OF SYMBOLS (concluded)

j	a specific generalised co-ordinate (j = 1, 2, ..., N)
k	a specific discrete lump (k = 1, 2, ..., n)
ℓ	discrete-lump axes directions
L	discrete lump
n	the n^{th} discrete lump
N	the N^{th} generalised co-ordinate
o	earth axes directions
s	structural
W	whole-body

Superscripts

b	undeformed-body axes
B	point occupied by the reference point in the undeformed-body state
d	datum-motion axes
D	point occupied by the reference point in the datum state
ℓ	discrete-lump axes
L	point occupied by a discrete-lump reference point
o	earth axes
O	some point fixed in the earth
P	point occupied by a particle

Suprascripts

•	(dot) derivative with respect to time
–	(bar) indicator of the total value of a generalised co-ordinate, that is, the sum of the datum-motion value plus an increment
~	(tilde) transposition of a matrix
^	(circumflex) indicator of a diagonal matrix

Summations

$\sum_{LA}$	summation over all discrete lumps of the aircraft
$\sum_{PA}$	summation over all particles of the aircraft
$\sum_{PL}$	summation over all particles of a discrete lump

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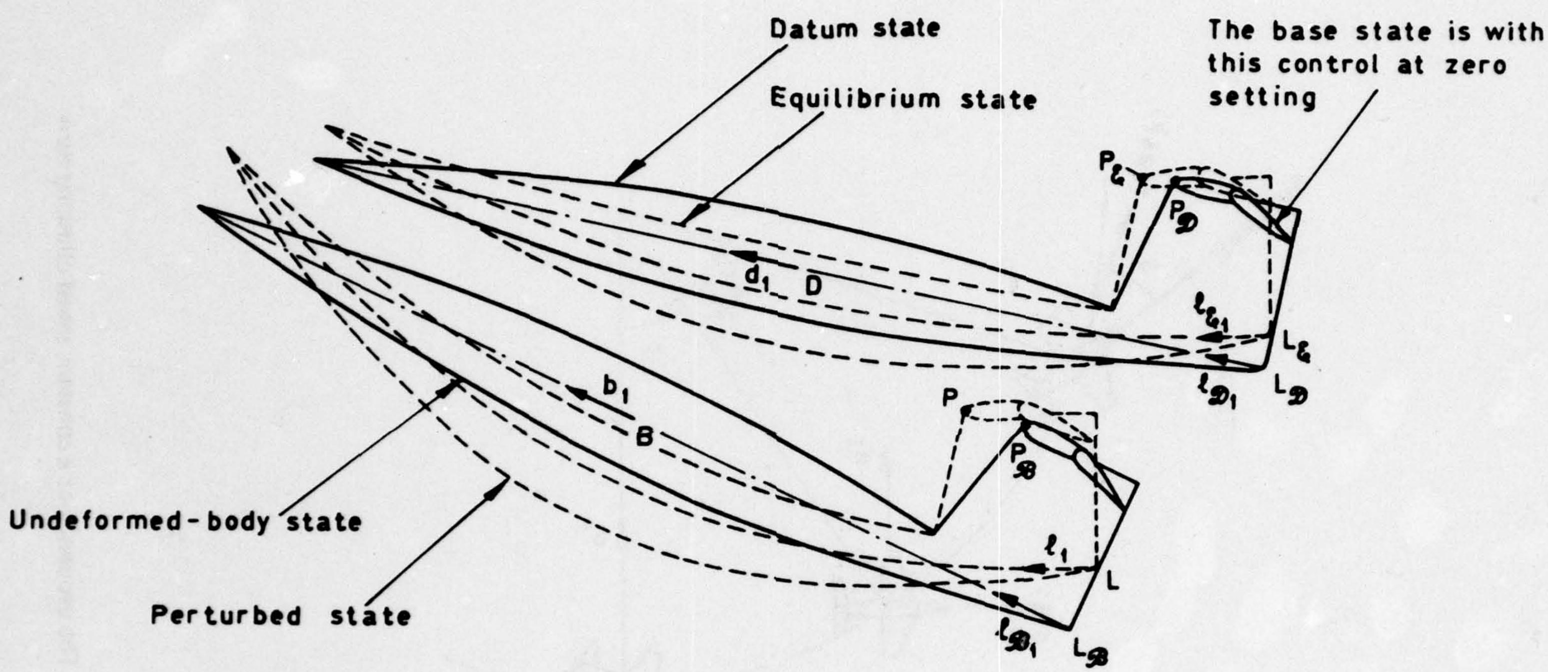

Fig 1 The five states of an aircraft

Fig 2

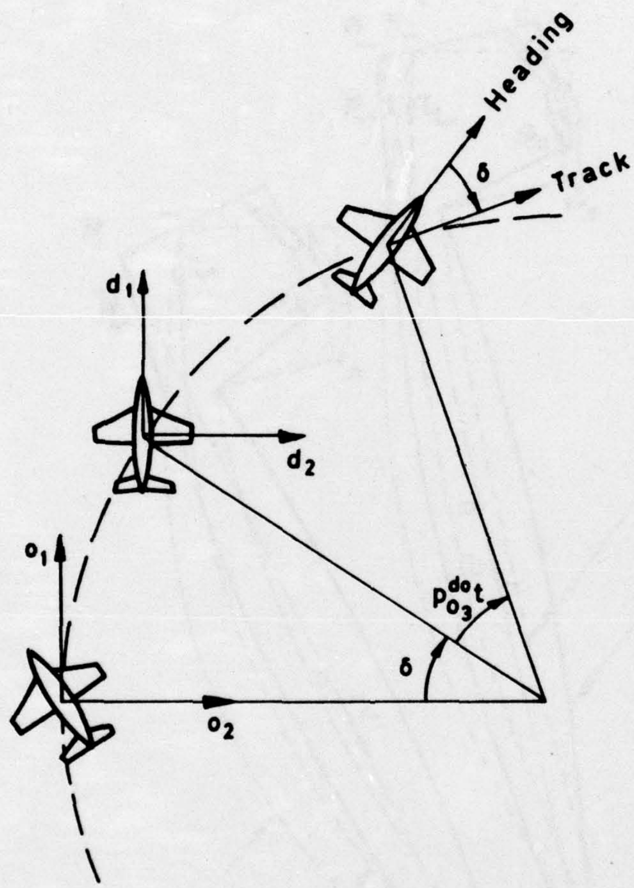

Fig 2 The geometry of a constant speed climbing turn

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